PROOF LAB I: DIRECT PROOF - HINTS

I. ($\star$) Show that for positive real numbers x and y, we have $x > y$ if and only if $x^2 > y^2$.

HINT(S):
- Compare x^2 and y^2 to xy using the fact that "If $a > 0$, then $b > c$ if and only if $ab > ac$".

II. ($\star$) Let x be an integer. Prove that x^n is odd if and only if x is odd.
(Hint: One direction uses the "binomial theorem", given above.)

HINT(S):
- Specifically, to prove "x is odd" $\Rightarrow$ "x^n is odd", you'll need the binomial theorem.
- To show something is odd, you need to write it as 2(some integer) + 1.
- I think it's easiest to show "x^n is odd" $\Rightarrow$ "x is odd" by showing the contrapositive. DOn't try to take an nth root!

III. ($\star\star$) Let $f : X \to Y$ be a function between two sets X and Y. Suppose A and B are subsets of X. Show that $f(A \cup B) = f(A) \cup f(B)$.

HINT(S):
- Start, for your own benefit, by writing the (set) definition of $f(A \cup B)$, $f(A)$, $f(B)$, and $f(A) \cup f(B)$.
- Go back to lecture 9, and use the set equality proof technique. Namely, start with "Let $x \in f(A \cup B)$. Then.... So $x \in f(A) \cup f(B)$." Then vice versa.

IV. ($\star\star$) Let X be a set, and let A and B be subsets of X. Prove $B - A = (A \cup B) - A$.

HINT(S):
- Again, go back to the lecture 9 notes, and use the set equality proof technique. Namely, start with "Let $x \in B - A$" and go from there.

V. ($\star\star\star$) Suppose x and y are rational numbers such that $x < y$. Prove that there exists $z \in \mathbb{Q}$ such that $x < z < y$. That is, between every two rationals there exists another rational.

HINT(S):
- If you use the "midpoint", $z = (x+y)/2$, you must prove that it is both rational and strictly between x and y. Try not to use "theorems" that we haven't proven.

VI. ($\star\star\star$) Prove that the following statements are equivalent for A and B subsets of X.

(i) $A \subseteq B$,

(ii) $A \cap B^c = \emptyset$,

(iii) $A^c \cup B = X$.

HINT(S):

- Try showing
  $(A \subseteq B) \Rightarrow (A \cap B^c = \emptyset)$
  ("Let $x \in A \cap B^c$. Then since $A \subset B$, ...);
  $(A \cap B^c = \emptyset) \Rightarrow (A^c \cup B = X)$;
  ("Let $x \in A^c \cup B$. Then since $A \cap B^c = \emptyset$, ...);
  and
  $(A^c \cup B = X) \Rightarrow (A \subseteq B)$
  ("Let $x \in A$. Then since $A^c \cup B = X$, ...);

VII. ($\star\star\star$) Prove that the following statements are equivalent.

(i) There exists unique x such that $P(x)$ is true.

(ii) There exists x such that (a) $P(x)$ holds and (b) for all y, we have $P(y) \Rightarrow y = x$.

(iii) There exists x so that for all y, $P(y)$ is true if and only if $y = x$.

(iv) There exists x such that $P(x)$ holds, and for all y and z, we have $(P(y) \wedge P(z)) \Rightarrow y = z$.

HINT(S):

- "There exists unique x such that $P(x)$ is true" means two things: there exists an x such that $P(x)$ is true; and there is not more than one x such that $P(x)$ is true.
- Most of these are just rewording the first statement—it's your job to make sure the first statement has been reworded correctly.

VIII. ($\star\star\star$) Let x_1, x_2, x_3, $\ldots x_n$ be an arrangement of the numbers $1, 2, 3, \ldots n$. (For example if $n = 5$ we could have $x_1 = 4, x_2 = 3, x_3 = 5, x_4 = 1, x_5 = 2$.) Prove that if n is odd, then $(x_1 - 1)(x_2 - 2)(x_3 - 3) \cdots (x_n - n)$ is even.
(Hint: Consider sums rather than products. You may use the fact that if the sum of some integers $n_1, \ldots, n_\ell$ is even, and exactly m of those are odd, then m is an even number.)

HINT(S):

- The hint above asks for you to notice that $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$. Now, subtract $1 + 2 + \cdots + n$ from both sides and rearrange so that you're saying something about the sum of $(x_i - i)$'s.