PROOF LAB III: CONTRADICTION

Instructions:

Step 1 (Today).
Work together with your group to outline the full set of exercises.

Step 2 (This weekend and next week).
Each person: pick 8 $\star$'s worth of exercises, and write up careful drafts of the proofs. Include key definitions and thorough reasoning—think about writing to convince a student taking Bridge next semester. Make sure a novice could follow and locate all critical components of the proof. **See portfolio guidelines for further instruction.**

Step 3 (over Thanksgiving weekend).
Swap drafts with a peer editor over email and give/get feedback. (Get a peer's email address and send them a pdf.)

Provide written constructive criticism for your partner: Are key definitions included? Is the reasoning mathematically accurate? Are the key components of the proof easy to recognize? Is the notation clear? Would a few additional comments help a novice reader to follow more easily? Can the explanation be made more clean and concise?

Step 4 (Thursday 11/29).
Turn in your proof second drafts, and comments from at least one peer editor on first drafts. (You will provide two versions of each proof–one first draft with comments, and one rewritten draft. Homework without peer editor comments will not be accepted.)

Problems. Prove each of the following using proof by contradiction.

I. ($\star$) For two integers a and b, assume that $4|(a^2+b^2)$. Then at least one of a or b is even.

II. ($\star$) Let $a,b,c \in \mathbb{Z}$. Assume that $\gcd(a,b)=1$ and $ab=c^2$. Then both a and b are squares of integers.

III. ($\star\,\star$) Let $\triangle ABC$ be a right triangle. Then at least one of the sides has either non-integer length or even-integer length.

[Hint: Start by drawing a picture, and writing a corresponding equation about the lengths of the triangle's sides. Then rewrite the statement "At least one of the sides has either non-integer length or even-integer length" as an if-then statement.]

IV. ($\star\,\star$) The none of the roots of $f(x)=x^3+x+1$ are rational.

V. ($\star\,\star$) Let n be a positive integer that is not prime (i.e. is *composite*). Then n has a prime divisor less than or equal to $\sqrt{n}$.

VI. Pick one:
($\star$) If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$.
($\star\,\star\,\star$) If x and y are non-zero real numbers, then $\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$.

VII. ($\star\star\star$) Let $A \subseteq \mathbb{R}$. We say A is *dense* in $\mathbb{R}$ if for every open interval $(a,b)=\{x\in\mathbb{R} \mid a<x<b\}$ in $\mathbb{R}$, there is at least one element of A in (a,b), i.e. $(a,b)\cap A \neq \emptyset$.

Claim: The set of rational numbers $\mathbb{Q}$ and the set of irrational numbers $X=\mathbb{R}-\mathbb{Q}$ are both dense in $\mathbb{R}$.