MIDTERM 2 TOPICS

1. Know everything for Midterm 1 well enough to use it to do any of the following kinds of problems.
2. Approximate integration. Draw pictures!
   (a) Midpoint rule: (rectangles)

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(x_i)\Delta x, \quad \text{where } \Delta x = (b-a)/n,\ x_i = a - \frac{1}{2} + i\Delta x.$$

   Error:

$$|E_M| \leq K(b-a)^3/24n^2, \qquad \text{where } K \geq f''(x) \text{ over } [a,b].$$

   (b) Trapezoid rule: (trapezoids)

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right), \quad \text{where } \Delta x = (b-a)/n,\ x_i = a + i\Delta x.$$

   Error:

$$|E_T| \leq K(b-a)^3/12n^2, \qquad \text{where } K \geq f''(x) \text{ over } [a,b].$$

   (c) Simpson's rule: (parabolas, n even)

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right),$$

   where $\Delta x = (b-a)/n$, $x_i = a + i\Delta x$.
   Error:

$$|E_S| \leq K(b-a)^5/180n^4, \qquad \text{where } K \geq f^{(4)}(x) \text{ over } [a,b].$$

3. Improper integrals.
   Does it converge or not? If so, what to?
   (a) Rewrite the integral using limit(s):
      (i) The function is continuous, but one of the endpoints is $\pm\infty$:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx; \qquad \int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx.$$

      (ii) The function is continuous, but both the endpoints are $\pm\infty$: pick any point a, and break the integral up as

$$\int_{-\infty}^\infty f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx + \lim_{t\to\infty}\int_a^t f(x)dx.$$

      (iii) The function is not continuous at one of the endpoints:

$$\text{disc. at } b: \quad \int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx; \qquad \text{disc. at } a: \quad \int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx.$$

      (iv) Some combination of all of the above: break it up at each of the problem points into integrals with one bound that's ok, and one bound that you need to treat with a limit (like (b)).

(b) Comparison test:

$$\text{If } g(x) \le f(x) \le h(x) \text{ over } [a,b], \quad \text{then } G \le F \le H, \quad \text{where} \quad \begin{aligned} G &= \textstyle\int_a^b g(x)dx, \\ F &= \textstyle\int_a^b f(x)dx, \\ H &= \textstyle\int_a^b h(x)dx. \end{aligned}$$

If $G \to \infty$ or $H \to -\infty$, then F diverges.
If G and H converge, F converges.
Otherwise, this doesn't tell you anything of use.

4. Area, Volumes, Arc length.
   (a) Area:
   Vertical slices: $dA = h(x)\ dx$, with $h(x) = y_{\text{top}} - y_{\text{bot}}$;
   Horizontal slices: $dA = h(y)\ dy$, with $h(y) = y_{\text{right}} - y_{\text{left}}$.
   (b) Volume:
   Pick your slices, determine your variable with endpoints, write a formula for dV, calculate any terms in dV as functions of your variable. Draw lots of pictures.
   Typical slices are

$$\text{Cylinders: } dV = A(x)dx \text{ or } A(y)dy;$$

$$\text{Cylindrical shells: } dV = 2\pi r(x)h(x)dx \text{ or } 2\pi r(y)h(y)dy.$$

   Volumes of revolution:
   (i) First pick horizontal or vertical slices (don't worry about your axis of rotation yet; just which endpoints are going to be easier to deal with).
   (ii) If your slice is perpendicular to your axis of rotation, you get washers (or discs if $r_{\in} = 0$).
   If your slice is parallel to your axis of rotation, you get cylindrical shells.
   (iii) Height is the difference between two functions bounding your region.
   Radius is the distance from the axis of rotation to the slice (washers: axis to function; shells: axis to variable)
   (c) Arc length.

$$d\ell^2 = dx^2 + dy^2$$

   so that

$$\ell = \int_{x=a}^{b} \sqrt{1 + (dy/dx)^2}\ dx = \int_{y=c}^{d} \sqrt{1 + (dx/dy)^2}\ dy.$$

   Use whichever gives you an integral you can compute.
5. Work.

$$W = Fd, \qquad \text{where } F \text{ is force and } d \text{ is distance moved.}$$

   Use $F = kx$ for springs and $F = ma$ for everything else.
   You might need to draw pictures and use geometry to calculate F or d as a function of position x.
   Gravity: For SI units, $g = 9.8$ m/s^2. For US units, lbs means both mass and force of that mass

under gravity on earth.
Major problems:

(i) springs
(ii) rope
(iii) tanks of water
(iv) leaky bucket (three parts: bucket, rope, and leaking water)