Midterm 1 topics

1. Inverse functions
   (a) Given a function, compute its inverse.
   (b) Domain/range.
   (c) Graphs.
   (d) $\frac{d}{dx} f^{-1}(x) = 1/(f'(f^{-1}(x)))$.
2. Special functions.
   (a) Exponentials and logs: $\ln(x), \log_a(x), e^x, a^x$.
   Know: definitions, limits, graphs, derivatives.
   (b) Inverse trig functions.
   Know: definitions, domain/ranges, limits, graphs, derivatives.
   In particular, know how to derive the derivatives, including how to simplify expressions like $\sin(\cos^{-1}(x))$.
   (c) Hyperbolic functions.
   Know: definitions, identities, domain/ranges, limits, graphs, derivatives.
3. Exponential growth/decay.
   (a) Main examples: population growth, radioactive decay, heating/cooling.
   (b) Know how to set up and what the solutions look like.
   (c) Use general solutions to solve for specific solutions (solve for all the unknowns).
   (d) Long term behavior.
4. Limits.
   (a) Basic limits, using all our new special functions.
   (b) L'Hopital's rule.
   (c) Indeterminate forms, and how to transform each into ∞/∞ or $0/0$.
5. Integrals.
   (a) Basic integrals using all our new special functions.
   In particular, $\tan^{-1}(x)$ and how to complete the square to get into the $a/(1+u^2)$ form.
   (b) Integration by parts.
   In particular,
      (i) integrate inverse functions like $\tan^{-1}(x)$ by letting $f = \tan^{-1}(x)$ and $g' = 1$, and
      (ii) integrate things like $e^x \cos(x)$ by letting $I = \int e^x \cos(x)\, dx$, doing IBP a couple times, seeing I appear again, and solving algebraically for I. (Note: this is how you do $\int \sec^3(x)\, dx$, letting $f = \sec(x)$ and $g' = \sec^2(x)$.)
   (c) Trig integrals.
   In particular, all the relevant trig identities, such as

$$\cos^2(x) + \sin^2(x) = 1 \qquad \tan^2(x) + 1 = \sec^2(x) \qquad \cot^2(x) + 1 = \csc^2(x)$$

$$\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x)) \qquad \sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

   Special integrals you want to know/specifically know how to compute: $\int \sec(x)\, dx$, $\int \csc(x)\, dx$, $\int \sec^3(x)\, dx$, and $\int \csc^3(x)\, dx$.
   (d) Trig sub, using substitutions like

$$\begin{aligned} ax = \sin(u) \text{ when you see } 1 - (ax)^2 &\qquad \text{because of the identity } \cos^2(u) = 1 - \sin^2(u) \\ ax = \tan(u) \text{ when you see } 1 + (ax)^2 &\qquad \text{because of the identity } \sec^2(u) = 1 + \tan^2(u) \\ ax = \sec(u) \text{ when you see } (ax)^2 - 1 &\qquad \text{because of the identity } \tan^2(u) = \sec^2(u) - 1, \end{aligned}$$

first factoring out c if you see something like $c - x^2$, $c + x^2$, $x^2 - c$, i.e.

$$c - x^2 = c(1 - x^2/c) = c(1 - (x/\sqrt{c})^2).$$

(e) Partial fractions decomposition.
  (i) Know your decomposition forms;
  (ii) know how to solve for the unknowns;
  (iii) know how to use natural logs and arctangents to do the resulting integrals.

Midterm 2 topics

1. Approximate integration. Draw pictures!
   (a) Midpoint rule: (rectangles)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x, \quad \text{where } \Delta x = (b-a)/n,\ x_i = a - \frac{1}{2} + i\Delta x.$$

   (b) Trapezoid rule: (trapezoids)

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right), \qquad \text{where } \Delta x = (b-a)/n,\ x_i = a + i\Delta x.$$

   (c) Simpson's rule: (parabolas, n even)

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right),$$

   where $\Delta x = (b-a)/n$, $x_i = a + i\Delta x$.

2. Improper integrals.
   Does it converge or not? If so, what to?
   (a) Rewrite the integral using limit(s):
     (i) The function is continuous, but one of the endpoints is $\pm\infty$:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx; \qquad \int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx.$$

     (ii) The function is continuous, but both the endpoints are $\pm\infty$: pick any point a, and break the integral up as

$$\int_{-\infty}^\infty f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx + \lim_{t\to\infty}\int_a^t f(x)dx.$$

     (iii) The function is not continuous at one of the endpoints:

$$\text{disc. at } b\text{:} \quad \int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx; \qquad \text{disc. at } a\text{:} \quad \int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx.$$

     (iv) Some combination of all of the above: break it up at each of the problem points into integrals with one bound that's ok, and one bound that you need to treat with a limit (like (b)).

(b) Comparison test:

$$\text{If } g(x) \le f(x) \le h(x) \text{ over } [a,b], \quad \text{then } G \le F \le H, \quad \text{where} \quad \begin{aligned} G &= \textstyle\int_a^b g(x)dx, \\ F &= \textstyle\int_a^b f(x)dx, \\ H &= \textstyle\int_a^b h(x)dx. \end{aligned}$$

If $G \to \infty$ or $H \to -\infty$, then F diverges.
If G and H converge, F converges.
Otherwise, this doesn't tell you anything of use.

3. Area, Volumes, Arc length.
   (a) Area:
   Vertical slices: $dA = h(x)\ dx$, with $h(x) = y_{\text{top}} - y_{\text{bot}}$;
   Horizontal slices: $dA = h(y)\ dy$, with $h(y) = y_{\text{right}} - y_{\text{left}}$.
   (b) Volume:
   Pick your slices, determine your variable with endpoints, write a formula for dV, calculate any terms in dV as functions of your variable. Draw lots of pictures.
   Typical slices are

$$\text{Cylinders: } dV = A(x)dx \text{ or } A(y)dy;$$

$$\text{Cylindrical shells: } dV = 2\pi r(x)h(x)dx \text{ or } 2\pi r(y)h(y)dy.$$

   Volumes of revolution:
   (i) First pick horizontal or vertical slices (don't worry about your axis of rotation yet; just which endpoints are going to be easier to deal with).
   (ii) If your slice is perpendicular to your axis of rotation, you get washers (or discs if $r_{\in} = 0$).
   If your slice is parallel to your axis of rotation, you get cylindrical shells.

   (iii) Height is the difference between two functions bounding your region.
   Radius is the distance from the axis of rotation to the slice (washers: axis to function; shells: axis to variable)
   (c) Arc length.

$$d\ell^2 = dx^2 + dy^2$$

   so that

$$\ell = \int_{x=a}^{b} \sqrt{1 + (dy/dx)^2}\ dx = \int_{y=c}^{d} \sqrt{1 + (dx/dy)^2}\ dy.$$

   Use whichever gives you an integral you can compute.

4. Work.

$$W = Fd, \qquad \text{where } F \text{ is force and } d \text{ is distance moved.}$$

   Use $F = kx$ for springs and $F = ma$ for everything else.
   You might need to draw pictures and use geometry to calculate F or d as a function of position x.
   Gravity: For SI units, $g = 9.8$ m/s^2. For US units, lbs means both mass and force of that mass

under gravity on earth.
Major problems:
(i) springs
(ii) rope
(iii) tanks of water
(iv) leaky bucket (three parts: bucket, rope, and leaking water)

The Rest

1. Parametric curves.
   (a) Graph sketching
   (b) Derivatives: $dy/dx = (dy/dt)/(dx/dt)$ and $d^2y/dx^2 = (\frac{d}{dt}(dy/dx))/(dx/dt)$.
   (c) Arc length: $d\ell^2 = dx^2 + dy^2$ so that
$$\ell = \int_{t=a}^{b} \sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt.$$
2. Polar curves.
   (a) Graph sketching. Basic graphs: circles, lines, cardioids, flowers.
   (b) Coordinate conversion:
$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$
   (c) Derivatives: $dy/dx = (dy/d\theta)/(dx/d\theta)$ and $d^2y/dx^2 = (\frac{d}{d\theta}(dy/dx))/(dx/d\theta)$, where $x = r\cos(\theta)$, $y = r\sin(\theta)$.
   (d) Arc length: $d\ell^2 = dx^2 + dy^2$ so that
$$\ell = \int_{\theta=a}^{b} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\ d\theta.$$
   (e) Areas:
$$A = \int_a^b \tfrac{1}{2} r^2\ d\theta$$
$$A = \int_a^b \tfrac{1}{2}(r_{\text{out}}^2 - r_{\text{in}}^2)\ d\theta$$
3. Conic sections.
   (a) Classifying $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$:

type	condition
parabola	$B^2 - 4AC = 0$
ellipse	$B^2 - 4AC < 0$
hyperbola	$B^2 - 4AC > 0$

   (i) If $B = 0$:
   - Parabola:

canonical form	vertex	focus	directrix	axis
$(x - x_0)^2 = 4p(y - y_0)$	(x_0, y_0)	$(x_0, y_0 + p)$	$y = y_0 - p$	$x = x_0$
$(y - y_0)^2 = 4p(x - x_0)$	(x_0, y_0)	$(x_0 + p, y_0)$	$x = x_0 - p$	$y = y_0$

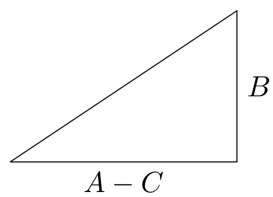
   - Ellipse:

canonical form	center	vertices	foci	axis
$\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 = 1$ where $a > b$	(x_0, y_0)	$(x_0 \pm a, y_0)$	$(x_0 \pm c, y_0)$ where $c^2 = a^2 - b^2$	$x = x_0$
$\left(\frac{x-x_0}{b}\right)^2 + \left(\frac{y-y_0}{a}\right)^2 = 1$ where $a > b$	(x_0, y_0)	$(x_0, y_0 \pm a)$	$(x_0, y_0 \pm c)$ where $c^2 = a^2 - b^2$	$y = y_0$

- Hyperbola:

canonical form	center	vertices	foci	asymptotes
$\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1$	(x_0, y_0)	$(x_0 \pm a, y_0)$	$(x_0 \pm c, y_0)$ where $c^2 = a^2 + b^2$	$y - y_0 = \pm(b/a)(x - x_0)$
$\left(\frac{y-y_0}{a}\right)^2 - \left(\frac{x-x_0}{b}\right)^2 = 1$	(x_0, y_0)	$(x_0, y_0 \pm a)$	$(x_0, y_0 \pm c)$ where $c^2 = a^2 + b^2$	$y - y_0 = \pm(a/b)(x - x_0)$

(ii) If $B \neq 0$:
Rotate the coordinate system by θ, where $\tan(\theta) = B/(A - C)$.
Use the triangle



to calculate $\cos(2\theta)$, and then

$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta)) \quad \text{and} \quad \sin^2(\theta) = \tfrac{1}{2}(1 - \cos(2\theta))$$

to calculate $\cos(\theta)$ and $\sin(\theta)$, where $\cos(\theta)$ is positive and $\sin(\theta)$ is the same sign as $\tan(2\theta)$.
Convert coordinates via

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta) \quad \text{and} \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta),$$
$$\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta).$$