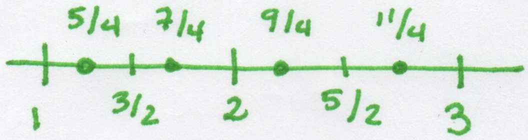
1. (15 pts) Consider $I = \int_1^3 \sin(5x)\,dx$.

(Pretend, for the purpose of this problem, that you don't know how to compute I exactly.)

(a) Use the midpoint rule to approximate I using 4 equal subintervals.



$$\Delta x = \frac{3-1}{4} = 1/2$$

$$M_4 = \Delta x \sum_{i=1}^{4} f(x_i)$$

$$= \frac{1}{2}\Big(\sin(5(5/4)) + \sin(5(7/4)) + \sin(5(9/4)) + \sin(5(11/4))\Big).$$

(b) Find an upper bound for the error of your approximation in part (a).

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \boxed{\frac{K\cdot 2^3}{24\cdot 4^2}},$$ where

where $K \geq |f''(x)|$ over $[1,3]$.

So since

$f'(x) = 5\cos(5x)$, $f''(x) = -25\sin(5x)$,

and $[1,3]$ contains a full period of $\sin(5x)$ $\left(2 > 2(\pi/5)\right)$,

$$\underset{(\text{over } [1,3])}{\max} |f''(x)| = 25\cdot 1. \qquad \boxed{K = 25}$$

2. (15 pts) Calculate the length of the curve $y = 1 - 2x^{3/2}$ from $x = 0$ to $x = 1$.

$$\ell = \int_{x=a}^{b} d\ell \qquad \text{where} \qquad d\ell = \sqrt{1 + (dy/dx)^2}\, dx$$

$$\frac{dy}{dx} = -2\left(\frac{3}{2}x^{1/2}\right) = -3x^{1/2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9x$$

$$\ell = \int_0^1 (1+9x)^{1/2}\, dx$$

let $u = 1 + 9x$
$du = 9du$

when $x = 0$, $u = 1 + 9\cdot 0 = 1$
when $x = 1$, $u = 1 + 9 \cdot 1 = 10$

$$= \int_1^{10} (1/9) u^{1/2}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \boxed{\frac{2}{9 \cdot 3}\left(10^{3/2} - 1^{3/2}\right)}$$

3. (20 pts)

(a) Preliminaries:

(i) Compute $\int \frac{1}{(1-x)^{1/3}}\,dx$.

Let $u = 1-x$, $du = -dx$

$$I = -\int u^{-1/3}du = -u^{2/3}\cdot\frac{3}{2} + C$$

$$= -(1-x)^{2/3}\cdot\frac{3}{2} + C$$

(ii) What is the domain of $\frac{1}{(1-x)^{1/3}}$?

All $x \neq 1$.

(iii) For what x is $\frac{1}{(1-x)^{1/3}}$ positive? For what x is $\frac{1}{(1-x)^{1/3}}$ negative?

$+$ | $-$

Check: @ 2: $\frac{1}{(1-2)^{1/3}} < 0$.

@ 0: $\frac{1}{(1-0)^{1/3}} > 0$.

(b) For each of the following:

- Decide whether the integral converges or diverges.
- If it converges, if possible, give the value that it converges to; if you had to use a comparison test to show convergence, give upper and lower bounds for the integral.

You may use work in any of the parts you've done to complete other parts if useful.

(i) $\displaystyle\int_0^1 \frac{1}{(1-x)^{1/3}}\,dx$ bad @ 1

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{(1-x)^{1/3}}\,dx$$

$$= \lim_{t\to 1^-} -\frac{3}{2}(1-x)^{2/3}\Big|_{x=0}^{t}$$

$$= -\frac{3}{2}\lim_{t\to 1^-} (1-t)^{2/3} - 1^{2/3}$$

$$= -3/2\,(0-1) = \boxed{3/2} \quad \text{(conv.)}$$

Part 2(c) continued...

(ii) $\displaystyle\int_0^2 \frac{1}{(1-x)^{1/3}}\,dx$ bad @ 1

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{(1-x)^{1/3}}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{1}{(1-x)^{1/3}}\,dx$$

$$= \frac{3}{2} + \lim_{s\to 1^+} -\frac{3}{2}(1-x)^{2/3}\Big|_s^2$$

$$= 3/2 + \lim_{s\to 1^+} -\frac{3}{2}\left((-1)^{2/3} - (1-s)^{2/3}\right)$$

$$= 3/2 - 3/2(1-0) = \boxed{0} \quad \text{(conv.)}$$

(iii) $\displaystyle\int_0^1 \frac{\cos^2(x)}{(1-x)^{1/3}}\,dx$

$0 \le \cos^2(x) \le 1$, so $0 \le \dfrac{\cos^2(x)}{(1-x)^{1/3}} \le \dfrac{1}{(1-x)^{1/3}}$, since $\dfrac{1}{(1-x)^{1/3}} \geq 0$ on $[0,1]$.

So

$$0 \le \int_0^1 \frac{\cos^2(x)}{(1-x)^{1/3}}\,dx \le \int_0^1 \frac{1}{(1-x)^{1/3}}\,dx = 3/2.$$

$\boxed{\text{converges between } 0 \text{ \& } 3/2.}$

(iv) $\displaystyle\int_2^\infty \frac{1}{(1-x)^{1/3}}\,dx$ bad @ ∞

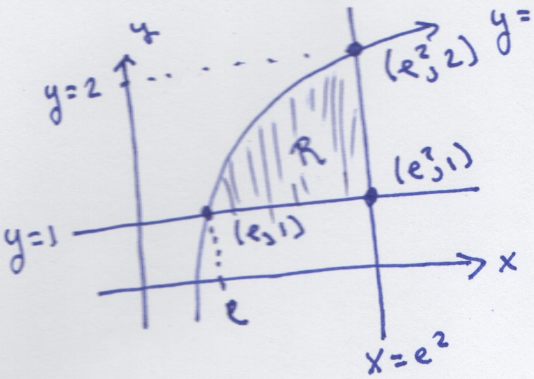
$$= \lim_{t\to\infty} \int_2^t \frac{1}{(1-x)^{1/3}}\,dx$$

$$= \lim_{t\to\infty} -\frac{3}{2}(1-x)^{2/3}\Big|_2^t$$

$$= -\frac{3}{2} \lim_{t\to\infty} \underbrace{(1-t)^{2/3}}_{\downarrow\, -\infty} - \underbrace{(-1)^{2/3}}_{\downarrow\, 1} = \infty \quad \boxed{\text{diverges}}$$

4. (35 pts) Consider the region $\mathcal{R}$ bounded by $y = \ln(x)$, the line $y = 1$, and the line $x = e^2$.

(a) Sketch this region, and find the x and y coordinates of each of the intersection points.



$y = \ln(x) \leftrightarrow x = e^y$

$\ln(e^2) = 2$

$\ln(e) = 1$

(b) Find the area of this region.

$$A = \int_{x=e}^{e^2} \text{top} - \text{bot}\ dx$$

$$= \int_e^{e^2} \ln(x) - 1\ dx$$

$$= x\ln(x) - x - x \Big|_e^{e^2}$$

$$= \boxed{e^2 \ln(e^2) - 2e^2 - (e\ln(e) - 2e)}$$

$$= \boxed{e}$$

$\int \ln(x)\, dx$

IBP: $f = \ln(x)$ $g' = 1$

$f' = 1/x$ $g = x$

$= x\ln(x) - \int x/x\, dx$

$= x\ln(x) - x + C$

or

$$A = \int_{y=1}^{2} \text{right} - \text{left}\ dy$$

$$= \int_1^2 e^2 - e^y\ dy$$

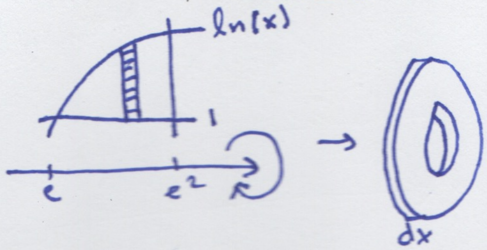
$$= e^2 y - e^y \Big|_1^2$$

$$= \boxed{e^2 \cdot 2 - e^2 - (e^2 - e)} = \boxed{e}$$

Continuing with $\mathcal{R}$ bounded by $y = \ln(x)$, the line $y = 1$, and the line $x = e^2$ as before.

(c) Consider the solid $\mathcal{S}$ generated by rotating $\mathcal{R}$ around the x-axis.

(i) Set up (but do not compute) an integral that calculates the volume of $\mathcal{S}$ using the washer/disc method.



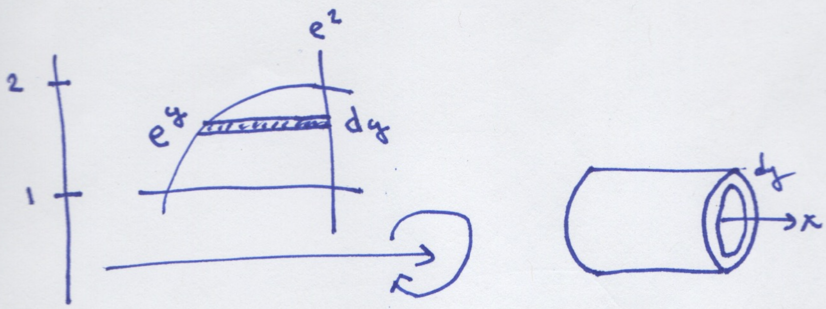
$dV = \pi(r_{out}^2 - r_{in}^2)\,dx$

$r_{out} = \ln(x)$

$r_{in} = 1$

$$\boxed{V = \int_e^{e^2} \pi\left(\ln^2(x) - 1^2\right)dx}$$

(ii) Set up (but do not compute) an integral that calculates the volume of $\mathcal{S}$ using the cylindrical shell method.



$dV = 2\pi r h\,dy$

$r = y$

$h = e^2 - e^y$

$$\boxed{V = \int_1^2 2\pi\, y\left(e^2 - e^y\right)dy}$$

Continuing with $\mathcal{R}$ the region bounded by $y = \ln(x)$, the line $y = 1$, and the line $x = e^2$, and $\mathcal{S}$ the solid generated by rotating $\mathcal{R}$ around the x-axis as before.

(iii) Use whichever of the previous parts you prefer to calculate the volume of $\mathcal{S}$.

$$V = \int_e^{e^2} \pi(\ln^2(x) - 1)\,dx$$

$$= \pi x(\ln^2(x) - 2\ln(x) + 1)\Big|_e^{e^2}$$

$$= \boxed{\pi e^2((2)^2 - 2(2) + 1) - \pi e(1^2 - 2\cdot 1 + 1)}$$

$$= \pi e^2$$

or

$$V = \int_1^2 2\pi y(e^2 - e^y)\,dy$$

$$= 2\pi\left(\frac{y^2}{2}e^2 - ye^y + e^y\right)\Big|_1^2$$

$$= \boxed{2\pi\left(\frac{2^2}{2}e^2 - 2e^2 + e^2 - \left(\frac{1}{2}e^y - e + e\right)\right)}$$

$$= \pi e^2 \checkmark$$

$$\int \ln^2(x)\,dx$$

$f = \ln^2(x)$ $\quad g' = 1$

$f' = 2\ln(x)\cdot\frac{1}{x}$ $\quad g = x$

$$= x\ln^2(x) - \int 2\ln(x)\cdot\frac{x}{x}\,dx$$

$$= x\ln^2(x) - (2)(x\ln(x) - x)$$

$$= x(\ln^2(x) - 2\ln(x) + 2) + c$$

so

$$\int \ln^2(x) - 1\,dx$$

$$= x(\ln^2(x) - 2\ln(x) + 1) + c$$

$$\int ye^y\,dy$$

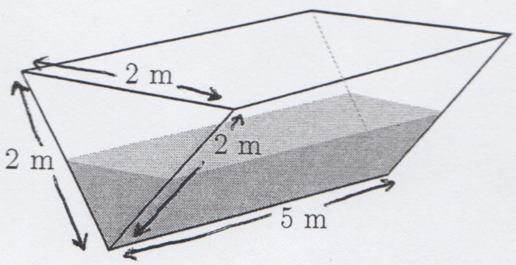
$f = y$ $\quad g' = e^y$

$f' = 1$ $\quad g = e^y$

$$= ye^y - \int e^y\,dy$$

$$= ye^y - e^y + c$$

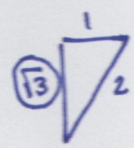
5. (15 pts) Consider a tank the shape of a triangular cylinder on its side, as shown, where the triangle is equilateral and 2m long on each side, and the length of the tank is 5 m. Suppose the tank is filled half way up with water (the depth of the water is half the depth of the tank). The density of water is 1000 kg/m^3 and the acceleration of gravity is 9.8 m/s^2.



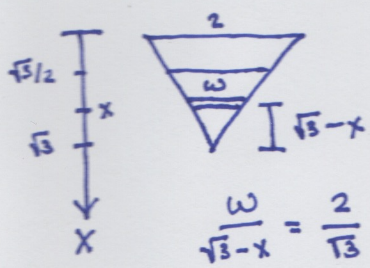
Set up (but do not compute) an integral for the work needed to pump the water out of the tank over the top. (Assume the water is always being pumped from the top of the water level, as usual.)

Hint. To get full credit, you definitely want to include:
(i) How tall is the tank? How deep is the water?
(ii) What is your coordinate system? Namely, where is your x-axis, and where is $x = 0$ and any other important coordinates?
(iii) How are you slicing this problem up, and what are the slices? How do you find the dimensions?
(iv) What is the formula for work?

tank height: $\sqrt{3}$



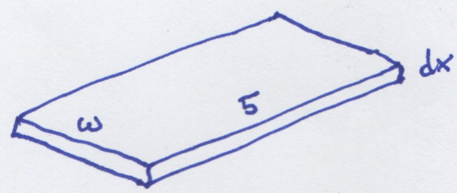
water height: $\sqrt{3}/2$



$$\frac{w}{\sqrt{3}-x} = \frac{2}{\sqrt{3}}$$

$$w = \frac{2}{\sqrt{3}}(\sqrt{3}-x)$$

Slices



$$V = 5w\,dx$$
$$= 5\left(\frac{2}{\sqrt{3}}\right)(\sqrt{3}-x)\,dx$$

$M = V \cdot \rho$, $\rho = 1000$

$F = Ma$, $a = 9.8$

$W = Fd = Fx$

$$\boxed{W = \int_{\sqrt{3}/2}^{\sqrt{3}} 1000)(9.8)(5)(2/\sqrt{3})(\sqrt{3}-x)\,x\,dx}$$