1. (15pts) Basic functions.

(a) For each of the following functions, give its derivative (versus x) and its domain. No need to show any work.

(i) $\log_3(x)$ $= \ln(x)/\ln(3)$

Derivative:	Domain:
$1/x\ln(3)$	all $x>0$

(ii) 3^x $= e^{\ln(3)\cdot x}$

Derivative:	Domain:
$\ln(3)\cdot 3^x$	all x

(iii) $\sin^{-1}(x)$

Derivative:	Domain:
$\frac{1}{\sqrt{1-x^2}}$	$-1\leq x\leq 1$ (range of $\sin(y)$)

(iv) $\cosh(x)$ $= \frac{1}{2}(e^x+e^{-x})$

Derivative:	Domain:
$\sinh(x)$	all x

(b) Evaluate the following limits. No need to show any work.

(ii) $\lim_{x\to-\infty} \sinh(x)$: $= \frac{1}{2}(e^x-e^{-x})$

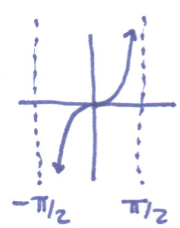
$\boxed{-\infty}$

since $e^x \to 0$
$e^{-x} \to \infty$

(i) $\lim_{x\to\infty} \tan^{-1}(x)$:

$\boxed{\pi/2}$

$\tan(y)$:



2. (40 pts) Integration.

(a) Compute $\int_0^{\pi/2} \sin^2(x)\cos^2(x)\,dx$. (Definite integral!)

$$\sin^2(x)\cos^2(x)$$

$$= \tfrac{1}{2}(1-\cos(2x)) \cdot \tfrac{1}{2}(1+\cos(2x))$$

$$= \frac{1}{4}\left(1-\cos^2(2x)\right)$$

$$= \frac{1}{4} - \frac{1}{4}\,\frac{1}{2}\left(1-\cos(4x)\right)$$

$$= \frac{1}{4} - \frac{1}{8} + \frac{1}{8}\cos(4x)$$

$$= \frac{1}{8} + \frac{1}{8}\cos(4x).$$

So $$I = \int_0^{\pi/2} \frac{1}{8} + \frac{1}{8}\cos(4x)\,dx$$

$$= \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin(4x)\Big|_0^{\pi/2}$$

$$= \frac{1}{8}(\pi/2) + \frac{1}{32}\left(\sin(2\pi) - \sin(0)\right)$$

$$= \boxed{\pi/16}$$

or

Similarly: $$\sin^2(x)\cos^2(x) = (\sin(x)\cos(x))^2$$

$$= \left(\tfrac{1}{2}\sin(2x)\right)^2$$

$$= \tfrac{1}{4}(1/2)(1-\cos(4x)).$$

(b) Compute $\int xe^{3x}\, dx$.

IBP: $\int f g' \, dx = fg - \int f' g \, dx$

$f = x \qquad g' = e^{3x}$

$f' = 1 \qquad g = \frac{1}{3} e^{3x}$

$$I = \frac{1}{3} x e^{3x} - \int 1 \cdot e^{3x} \left(\frac{1}{3}\right) dx$$

$$= \boxed{\frac{1}{3} x e^{3x} - \left(\frac{1}{3}\right)^2 e^{3x} + C}$$

(c) Compute $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$.

Trig sub: $\sec^2(u) - 1 = \tan^2(u)$ (*)

let $x = \sec(u)$,

so $dx = \sec(u)\tan(u)\,du$.

$$I = \int \frac{1}{\sec^2(u)\underbrace{\sqrt{\sec^2(u)-1}}_{(*)}} \cdot \sec(u)\tan(u)\,du$$

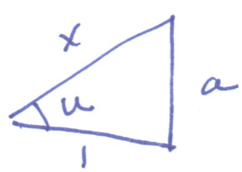
$$= \int \frac{\sec(u)\tan(u)}{\sec^2(u)\tan(u)}\,du$$

$$= \int 1/\sec(u)\,du$$

$$= \int \cos(u)\,du = \sin(u) + C$$

$$= \boxed{\sqrt{x^2-1}/x + C}$$

$\sec(u) = x/1$



$\rightarrow \sin(u) = \frac{\sqrt{x^2-1}}{x}$

$1 + a^2 = x^2$

$a = \sqrt{x^2-1}$

(d) Compute $\int \frac{5x^2 - x + 2}{x^3 + x}$.

P.F.

$$x^3 + x = (x^2+1)x$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{5x^2 - x + 2}{x(x^2+1)}$$

So

$$A(x^2+1) + (Bx+C)x = 5x^2 - x + 2$$

$$Ax^2 + A + Bx^2 + Cx =$$

$$x^2(A+B) + xC + A =$$

So

$A + B = 5 \qquad C = -1 \qquad A = 2 \longrightarrow B = 3$

$$I = \int \frac{2}{x} + \frac{3x-1}{x^2+1}\,dx.$$

- $\int \frac{2}{x}\,dx = \boxed{2\ln|x| + C}$
- $\int \frac{3x}{x^2+1}\,dx \qquad u = x^2+1, \quad du = 2x\,dx$

$= \frac{3}{2}\int \frac{1}{u}\,du = \boxed{\frac{3}{2}\ln|x^2+1| + C}$

- $\int \frac{-1}{x^2+1}\,dx = \boxed{-\tan^{-1}(x) + C}$

$$= \boxed{2\ln|x| + \frac{3}{2}\ln|x^2+1| - \tan^{-1}(x) + C}$$

3. (15 pts) Limits.

(a) Compute $\lim_{x\to\infty} e^{-x}\cos(x)$.

since

and $e^{-x} \longrightarrow 0$

$-1 \le \cos(x) \le 1$,

so $e^{-x}\cos(x) \longrightarrow \boxed{0}$.

(b) Compute $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$. $\begin{matrix}\longrightarrow 0 \\ \longrightarrow 0\end{matrix}$ indet!

$\overset{L'H}{=} \lim_{x\to 0} \frac{\sin(x)}{2x}$ $\begin{matrix}\to 0 \\ \to 0\end{matrix}$ indet!

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{2}$ $\begin{matrix}\to 1 \\ \to 2\end{matrix}$ $= \boxed{1/2}$

(c) Compute $\lim_{x\to\infty} x^{e^{-x}}$.

let $L = \lim_{x\to\infty} x^{e^{-x}}$.

So $\ln(L) = \lim_{x\to\infty} \ln(x^{e^{-x}})$

$= \lim_{x\to\infty} e^{-x} \ln(x)$ indet!

$\quad e^{-x} \downarrow 0 \qquad \ln(x) \downarrow \infty$

$= \lim_{x\to\infty} \frac{\ln(x)}{e^x} \begin{matrix} \to \infty \\ \to \infty \end{matrix}$ indet!

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{e^x} \begin{matrix} \to 0 \\ \to \infty \end{matrix} = \boxed{0}$

Since

$\ln(L) = 0$,

$L = e^0 = \boxed{1.}$

4. (15 pts) Suppose a population of rabbits grows at a rate proportional to its size. If the population doubles in a month, how long will it take (in months) for the population to grow to ten times its original size? Be sure to show all your work.

$$\frac{dy}{dt} = ky \quad , \quad \text{so} \quad y = y_0 e^{kt}$$

where $y_0 = y(0)$.

doubles:

$$\underbrace{2y_0 = y_0}_{\text{cancels}} e^{k \cdot 1}$$

so $2 = e^k$. Thus $k = \ln(2)$.

So

$$y = y_0 e^{\ln(2) \cdot t}.$$

t for $10 \times$ initial?

Solve

$$\underbrace{10 y_0 = y_0}_{\text{cancels}} e^{\ln(2) \cdot t}$$

$$10 = e^{\ln(2) t} \xrightarrow{\ln} \ln(10) = \ln(2) t$$

so $\boxed{t = \ln(10) / \ln(2)}$

5. (15 pts) Show why $\frac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$.

let $y = \tan^{-1}(x)$.

So $\tan(y) = x$.

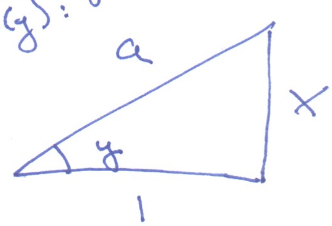
Thus

$$\frac{d}{dx}\tan(y) = \sec^2(y)\cdot\frac{dy}{dx}$$

$$= \frac{d}{dx}x = 1$$

so $\dfrac{dy}{dx} = 1/\sec^2(y)$.

calculating $\sec(y)$:



$a^2 = x^2 + 1^2$

$a = \sqrt{x^2+1}$

$\sec(y) = a/1$.

$$= 1/(a/1)^2$$

$$= 1/\left(\sqrt{x^2+1}\right)^2$$

$$= \boxed{1/1+x^2}$$

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