Warm up

Recall, to graph a conic function, you want it in the form

parabola: $(x-x_0)^2 = 4p(y-y_0)$ or $(y-y_0)^2 = 4p(x-x_0)$,

ellipse: $\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 = 1$,

hyperbola: $\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1$ or $\left(\frac{y-y_0}{a}\right)^2 - \left(\frac{x-x_0}{b}\right)^2 = 1$

Without doing any algebraic manipulation, decide whether each of the following conic sections is a parabola, ellipse, or hyperbola. Can you tell which way it's oriented?

1. $36y^2 - x^2 + 72y + 6x + 31 = 0$
2. $y^2 - 10y - 8x + 49 = 0$
3. $y^2 + 2x^2 + 4y + 8x + 11 = 0$

Now, complete squares and put into a form you can graph from. Identify (as applicable) the vertices, foci, axes, directrixes, asymptotes.

Conics with cross-terms

Again, all conic sections satisfy an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for some constants A, B, C, D, E, F.

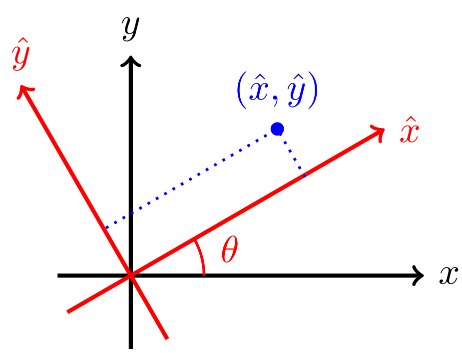
Last time, we talked about how to analyze anything where $B = 0$.

Just like we can shift up/down or left/right to get from something like

$$(y-5)^2 = 8(x-3) \qquad \text{to} \qquad (\hat{y})^2 = 8\hat{x},$$

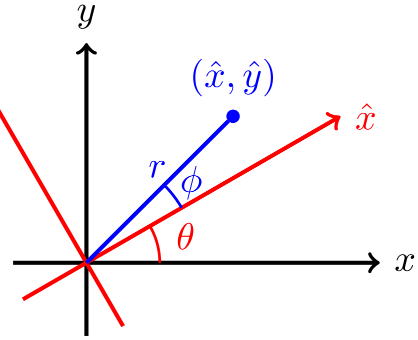
we can rotate to get from something where $B \neq 0$ to something where $B = 0$.

Take the usual x, y-axis, and rotate it by θ in $[0, \pi/2)$ to get a new $\hat{x}, \hat{y}$-axis:



We want to take a point $(\hat{x}, \hat{y})$ and write it in terms of x and y.

Take the usual x, y-axis, and rotate it by θ in $[0, \pi/2)$ to get a new $\hat{x}, \hat{y}$-axis:



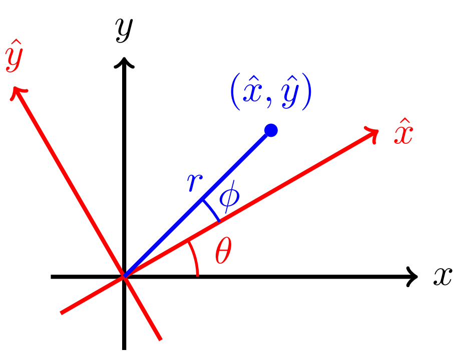
We want to take a point $(\hat{x}, \hat{y})$ and write it in terms of x and y.
Easier to do in polar!
In the $\hat{x}, \hat{y}$-frame,

$$\hat{x} = r\cos(\phi) \qquad \hat{y} = r\sin(\phi).$$

But in the x, y-frame, the angle is $\phi + \theta$, and the radius is the same! So in the x, y-frame,

$$x = r\cos(\phi + \theta) \qquad y = r\sin(\phi + \theta).$$

Take the usual x, y-axis, and rotate it by θ in $[0, \pi/2)$ to get a new $\hat{x}, \hat{y}$-axis:



$$\hat{x} = r\cos(\phi),$$
$$\hat{y} = r\sin(\phi)$$

$$x = r\cos(\phi + \theta)$$
$$y = r\sin(\phi + \theta)$$

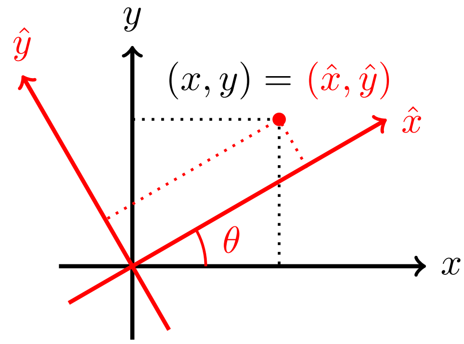
Now, using the angle addition formulas,

$$x = r\cos(\phi + \theta) = r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta) = \hat{x}\cos(\theta) - \hat{y}\sin(\theta),$$
$$y = r\sin(\phi + \theta) = r\cos(\phi)\sin(\theta) + r\sin(\phi)\cos(\theta) = \hat{x}\sin(\theta) + \hat{y}\cos(\theta).$$

Combining these equations, we get

$$x\cos(\theta) + y\sin(\theta) = \hat{x}(\cos^2(\theta) + \sin^2(\theta)) + \hat{y} \cdot 0 = \hat{x}, \text{ and}$$
$$-x\sin(\theta) + y\cos(\theta) = \hat{x} \cdot 0 + \hat{y}(\sin^2(\theta) + \cos^2(\theta)) = \hat{y}.$$

Change of coordinates for rotating axes:



If I rotate the x, y-axis by θ to get new coordinates $(\hat{x}, \hat{y})$, then the conversion between coordinate systems is given by

$$\boxed{x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta) \quad \text{and} \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)}$$

$$\boxed{\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta)}$$

Example

Show that the function $xy = 1$ is a hyperbola rotated by $\pi/4$. What are the foci and the asymptotes?

Solution. Convert coordinates, using $\theta = \pi/4$:

$$x = \hat{x}\cos(\pi/4) - \hat{y}\sin(\pi/4) = \tfrac{\sqrt{2}}{2}(\hat{x} - \hat{y}), \text{ and}$$
$$y = \hat{x}\sin(\pi/4) + \hat{y}\cos(\pi/4) = \tfrac{\sqrt{2}}{2}(\hat{x} + \hat{y}).$$

So, subbing into $xy = 1$, we get

$$1 = xy = \left(\tfrac{\sqrt{2}}{2}(\hat{x} - \hat{y})\right)\left(\tfrac{\sqrt{2}}{2}(\hat{x} + \hat{y})\right) = \tfrac{1}{2}(x^2 - y^2) = \left(\tfrac{\hat{x}}{\sqrt{2}}\right)^2 - \left(\tfrac{\hat{y}}{\sqrt{2}}\right)^2.$$

In the $\hat{x}, \hat{y}$ frame, the asymptotes are $\hat{y} = \pm\hat{x}$. But

$\hat{y} = \hat{x}$ is equivalent to $\hat{x} - \hat{y} = 0,$ which is equivalent to $x = 0$, and

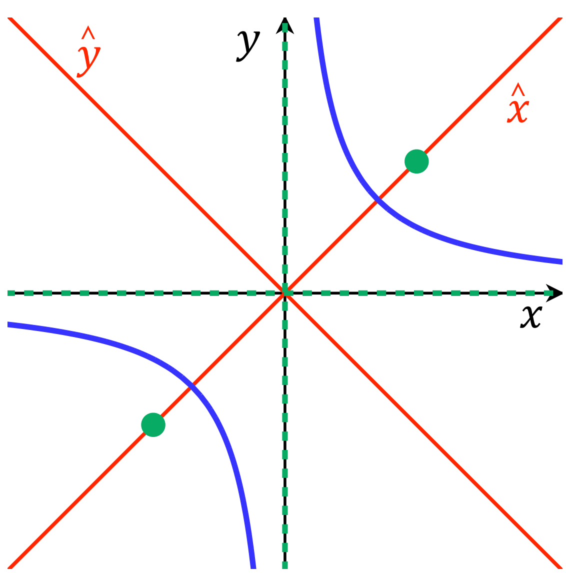
$\hat{y} = -\hat{x}$ is equivalent to $\hat{x} + \hat{y} = 0,$ is equivalent to $y = 0$.

Also, in the $\hat{x}, \hat{y}$ frame, the foci are $(\hat{x}, \hat{y}) = (\pm 2, 0)$. So

$$x = \pm 2\cos(\pi/4) + 0 \cdot \sin(\pi/4) = \pm\sqrt{2}, \text{ and}$$
$$y = \pm(2)\sin(\pi/4) + 0 \cdot \cos(\pi/4) = \pm\sqrt{2}.$$

Graphing $xy = 1$:

In summary, the function $xy = 1$ is the function $\left(\frac{\hat{x}}{\sqrt{2}}\right)^2 - \left(\frac{\hat{y}}{\sqrt{2}}\right)^2$ rotated by $\theta = \pi/4$. In the x, y-frame, the asymptotes are $x = 0$ and $y = 0$. The foci are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.



(Compare to $y = 1/x$!)

Hypothesis: for a conic section
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, there is some rotational change of coordinates in which the xy term is missing.

In the previous example, we were explicitly given the right angle to rotate by to eliminate the xy-term.

Big question: How could we find this in general?

Strategy:
Take a generic θ and plug the change of coordinates

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$

in to the generic conic section equation.
Collect "like terms" in $\hat{x}$ and $\hat{y}$.
The coefficient of $\hat{x}\hat{y}$ will be in terms of θ, A, B, etc.. Solve for θ so that, in this new equation, the coefficient of $\hat{x}\hat{y}$ is 0.

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$

$$\begin{aligned} 0 &= Ax^2 + Bxy + Cy^2 + Dx + Ey + F \\ &= A(\hat{x}\cos(\theta) - \hat{y}\sin(\theta))(\hat{x}\cos(\theta) - \hat{y}\sin(\theta)) \\ &\quad + B(\hat{x}\cos(\theta) - \hat{y}\sin(\theta))(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) \\ &\quad + C(\hat{x}\sin(\theta) + \hat{y}\cos(\theta))(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) \\ &\quad + D(\hat{x}\cos(\theta) - \hat{y}\sin(\theta)) + E(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) + F \end{aligned}$$

Coefficient of $\hat{x}\hat{y}$:

$$\begin{aligned} &-A\cos(\theta)\sin(\theta) - A\sin(\theta)\cos(\theta) \\ &\quad +B\cos(\theta)\cos(\theta) - B\sin(\theta)\sin(\theta) \\ &\quad +C\sin(\theta)\cos(\theta) + C\cos(\theta)\sin(\theta) \\ &= B(\cos^2(\theta) - \sin^2(\theta)) + (C - A)(2\cos(\theta)\sin(\theta)) \\ &= B\cos(2\theta) + (C - A)\sin(2\theta). \end{aligned}$$

So we want θ such that

$$B\cos(2\theta) + (C - A)\sin(2\theta) = 0, \text{ i.e. } \boxed{\tan(2\theta) = B/(A - C)}.$$

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$

Rotating by θ eliminates the xy terms when $\tan(2\theta) = B/(A - C)$.

What θ do you want to rotate by to graph each of the following conic sections?

Example: $8x^2 - 4xy + 4y^2 + 1 = 0$.

Solution: $A = 8$, $B = -4$, $C = 4$, so $\tan(2\theta) = 4/(8 - 4) = 1$. So $2\theta = \pi/4$. So $\theta = \pi/8$.

You try:

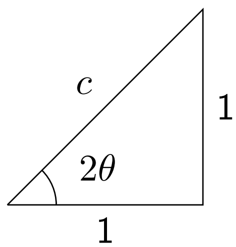
1. $x^2 - xy + y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$
$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta),\ y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta),\ \tan(2\theta) = B/(A-C).$$

Of course, plugging in things like $\theta = \pi/8$ into $\sin(\theta)$ and $\cos(\theta)$ isn't fun without a calculator. Draw a triangle for $\cos(2\theta)$ and use the half angle formulas!

$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta)) \quad \sin^2(\theta) = \tfrac{1}{2}(1 - \cos(2\theta))$$

Example: If $\tan(2\theta) = 1$, then 2θ is the angle of the triangle



$$c = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\cos(2\theta) = 1/\sqrt{2} = \sqrt{2}/2$$
$$\cos(\theta) = \sqrt{\tfrac{1}{2}(1 + \sqrt{2}/2)}$$
$$\sin(\theta) = \sqrt{\tfrac{1}{2}(1 - \sqrt{2}/2)}$$

So

$$x = \hat{x}\sqrt{\tfrac{1}{2}(1 + \tfrac{\sqrt{2}}{2})} - \hat{y}\sqrt{\tfrac{1}{2}(1 - \tfrac{\sqrt{2}}{2})}, \quad y = \hat{x}\sqrt{\tfrac{1}{2}(1 - \tfrac{\sqrt{2}}{2})} + \hat{y}\sqrt{\tfrac{1}{2}(1 + \tfrac{\sqrt{2}}{2})}$$

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$
$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$
$$\tan(2\theta) = B/(A-C).$$
$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta)) \quad \sin^2(\theta) = \tfrac{1}{2}(1 - \cos(2\theta))$$

You try: Find the change of coordinates that will eliminate the xy term in each of the following conic sections.

1. $x^2 - xy - y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

Identifying the conic section

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

In the case we did last time, where $B = 0$, we identified the cases

shape	condition if $B = 0$:	
parabola	$A = 0$ or $C = 0$	$AC = 0$
ellipse	A and C same signs	$AC > 0$
hyperbola	A and C different signs	$AC < 0$

Just like we did for calculating the coefficient of $\hat{x}\hat{y}$ after changing coordinates, we could calculate all the other coefficients after a change in coordinates

$$0 = \hat{A}(\hat{x})^2 + B\hat{x}\hat{y} + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F.$$

If we did, we could also show that the discriminants before and after the change are the same, i.e.

$$B^2 - 4AC = (\hat{B})^2 - 4\hat{A}\hat{C}.$$

But if we chose θ so that $\hat{B} = 0$ (like we've been doing), then we get

$$B^2 - 4AC = -4\hat{A}\hat{C}.$$

Putting it together: we start with

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

and convert to

$$0 = \hat{A}(\hat{x})^2 + B\hat{x}\hat{y} + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F,$$

using

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \; y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta).$$

We choose θ such that $\hat{B} = 0$, i.e. $\tan(2\theta) = B/(A - C)$.
Then since $B^2 - 4AC = (\hat{B})^2 - 4\hat{A}\hat{C}$, we get

$$-4\hat{A}\hat{C} = B^2 - 4AC.$$

Using our table from before, we can classify the conic section according to the following table:

shape	condition	
parabola	$\hat{A}\hat{C} = 0$	$B^2 - 4AC = 0$
ellipse	$\hat{A}\hat{C} > 0$	$B^2 - 4AC < 0$
hyperbola	$\hat{A}\hat{C} < 0$	$B^2 - 4AC > 0$

You try

Identify the kind of conic section given by each of the following equations:

1. $x^2 - xy - y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

Graphing $0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$

1. If $B \neq 0$, change coordinates using $x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta)$, $y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$, where $\tan(2\theta) = B/(A-C)$.
   Tip: Use a triangle and the half-angle formulas to calculate $\cos(\theta)$ and $\sin(\theta)$.
2. With your new equation
   $$0 = \hat{A}(\hat{x})^2 + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F,$$
   get into one of the canonical forms and calculate all the important features of the graph.
   (If $B = 0$, then $\hat{A} = A$, $\hat{C} = C$, etc..)
3. Sketch the graph by first sketching the rotated axes and then graphing the ˆ curve on those axes.
4. If needed, convert features of the graph (like vertices, foci, asymptotes, etc.) back into x, y-coordinates using
   $$\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta).$$

You try: Sketch a graph of $4x^2 - 4xy + 7y^2 - 24 = 0$.