Warm up

Recall, to graph a conic function, you want it in the form

parabola: $(x - x_0)^2 = 4p(y - y_0)$ or $(y - y_0)^2 = 4p(x - x_0)$,

ellipse: $\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 = 1$,

hyperbola: $\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1$ or $\left(\frac{y-y_0}{a}\right)^2 - \left(\frac{x-x_0}{b}\right)^2 = 1$

Without doing any algebraic manipulation, decide whether each of the following conic sections is a parabola, ellipse, or hyperbola. Can you tell which way it's oriented?

1. $36y^2 - x^2 + 72y + 6x + 31 = 0$
2. $y^2 - 10y - 8x + 49 = 0$
3. $y^2 + 2x^2 + 4y + 8x + 11 = 0$

Now, complete squares and put into a form you can graph from. Identify (as applicable) the vertices, foci, axes, directrixes, asymptotes.

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Without doing any algebraic manipulation, decide whether each of the following conic sections is a parabola, ellipse, or hyperbola. Can you tell which way it's oriented?

1. $36y^2 - x^2 + 72y + 6x + 31 = 0$ $\quad\left(\frac{x-3}{2}\right)^2 - \left(\frac{y+1}{1/3}\right)^2 = 1$
2. $y^2 - 10y - 8x + 49 = 0$ $\quad(y-5)^2 = 4(2)(x-3)$
3. $y^2 + 2x^2 + 4y + 8x + 11 = 0$ $\quad(y+2)^2 + \left(\frac{x+2}{1/\sqrt{2}}\right)^2 = 1$

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So $4 = (x-3)^2 - 36(y+1)^2$. So

$$1 = (x-3)^2/4 - 9(y+1)^2 = \left(\frac{x-3}{2}\right)^2 - \left(\frac{y+1}{1/3}\right)^2.$$

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center: $(3, -1)$; $a = 2$, $b = 1/3$,

$c = \sqrt{2^2 + (1/3)^2} = \sqrt{37}/3 \approx 2.03$; ;

vertices: $(3 \pm 2, -1)$;

foci: $(3 \pm c, -1)$; (note: very close to the vertices!)

asymptotes: $y + 1 = \pm \left(\frac{1/3}{2}\right)(x - 3)$

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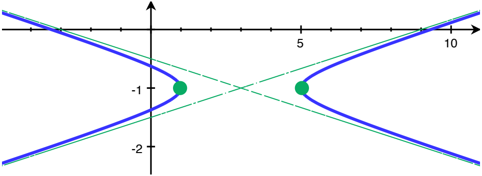

$y^2 - 10y - 8x + 49 = 0$: Sideways parabola! (opens to the right)

$$0 = y^2 - 10y - 8x + 49 = (y-5)^2 - 25 - 8x + 49 = (y-5)^2 - 8x + 24,$$

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$$(y-5)^2 = 8x - 24 = 8(x-3) = 4(2)(x-3).$$

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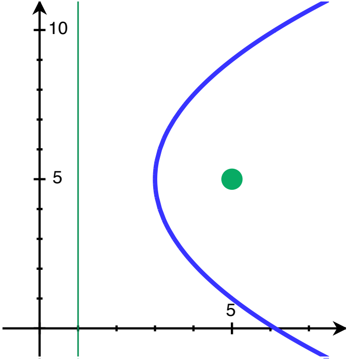

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$$1 = (y+2)^2 + 2(x+2)^2 = (y+2)^2 + \left(\frac{x+2}{1/\sqrt{2}}\right)^2.$$

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center: $(-2, -2)$;
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vertices: $(-2, -2 \pm a)$;
foci: $(-2, -2 \pm c)$;
major axis: $x = -2$

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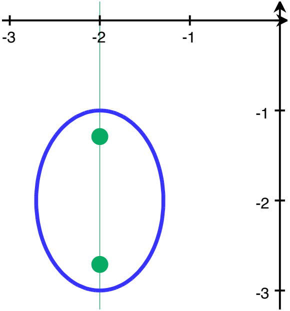

Conics with cross-terms

Again, all conic sections satisfy an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for some constants A, B, C, D, E, F.

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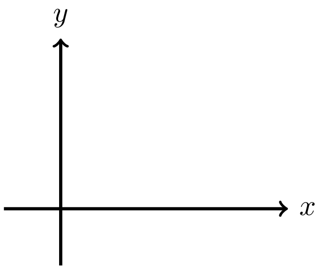
Last time, we talked about how to analyze anything where $B = 0$.

Just like we can shift up/down or left/right to get from something like

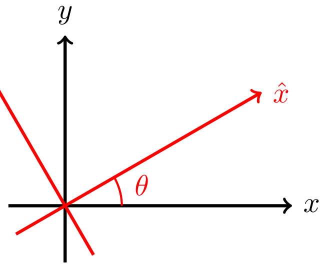
$$(y-5)^2 = 8(x-3) \qquad \text{to} \qquad (\hat{y})^2 = 8\hat{x},$$

we can rotate to get from something where $B \neq 0$ to something where $B = 0$.

Take the usual x, y-axis



Take the usual x, y-axis, and rotate it by θ in $[0, \pi/2)$ to get a new $\hat{x}, \hat{y}$-axis:



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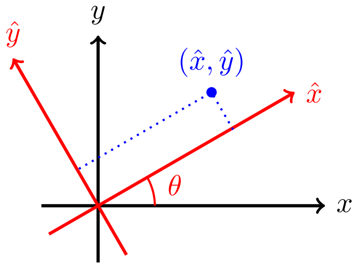

We want to take a point $(\hat{x}, \hat{y})$ and write it in terms of x and y.

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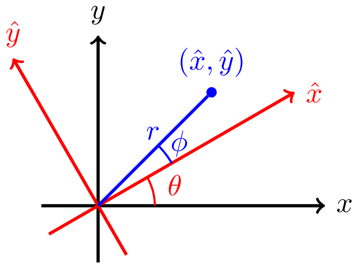

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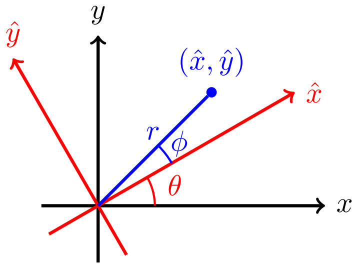

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$$\hat{x} = r\cos(\phi) \qquad \hat{y} = r\sin(\phi).$$

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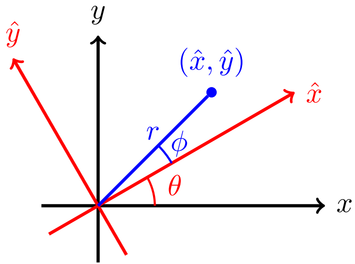

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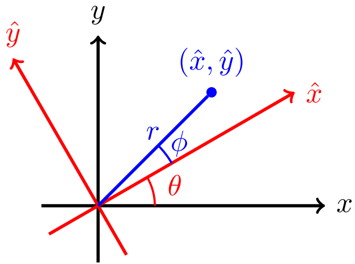

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$$x = r\cos(\phi + \theta) \qquad y = r\sin(\phi + \theta).$$

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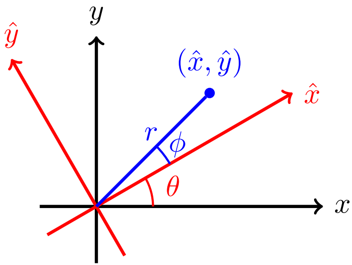

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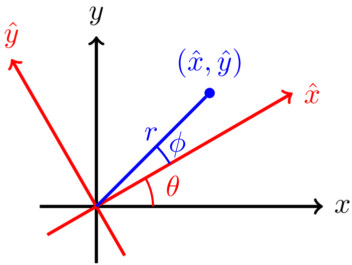

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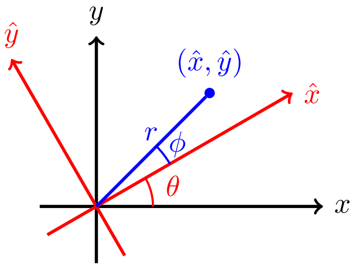

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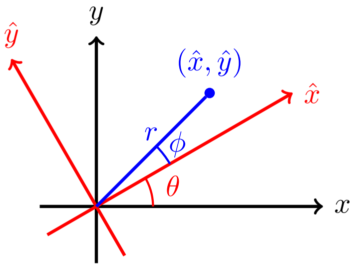

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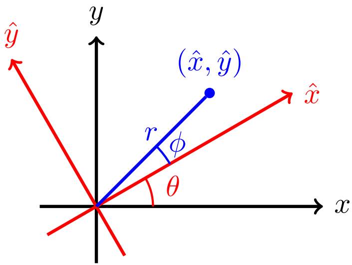

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Combining these equations, we get

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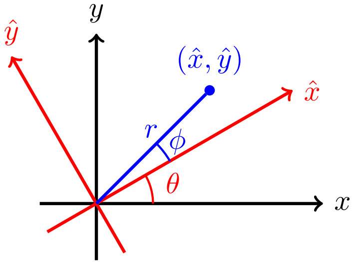

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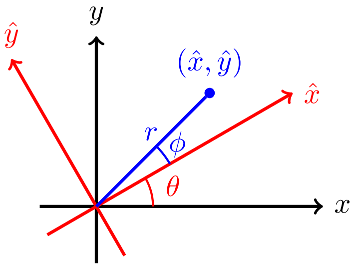

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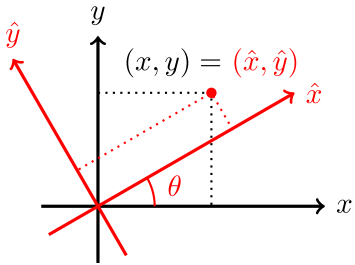
Change of coordinates for rotating axes:



If I rotate the x, y-axis by θ to get new coordinates $(\hat{x}, \hat{y})$, then the conversion between coordinate systems is given by

$$\boxed{x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta) \quad \text{and} \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)}$$

$$\boxed{\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta)}$$

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In summary, the function $xy = 1$ is the function $\left(\frac{\hat{x}}{\sqrt{2}}\right)^2 - \left(\frac{\hat{y}}{\sqrt{2}}\right)^2$ rotated by $\theta = \pi/4$. In the x, y-frame, the asymptotes are $x = 0$ and $y = 0$. The foci are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

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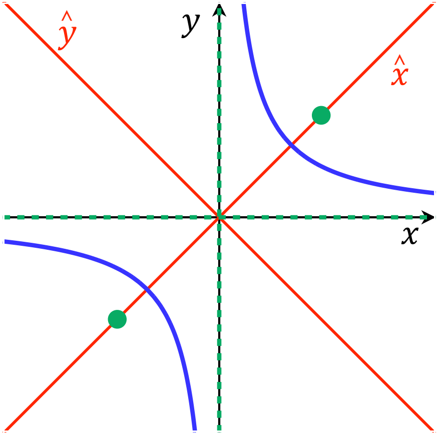

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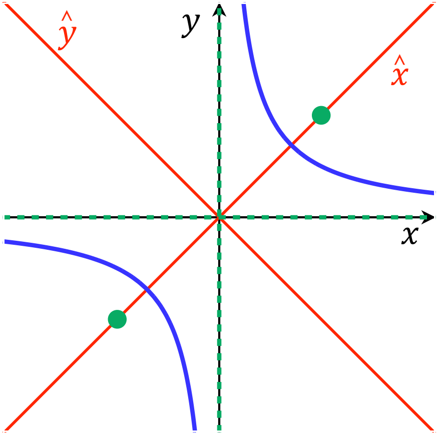

(Compare to $y = 1/x$!)

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Take a generic θ and plug the change of coordinates

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Collect "like terms" in $\hat{x}$ and $\hat{y}$.
The coefficient of $\hat{x}\hat{y}$ will be in terms of θ, A, B, etc.. Solve for θ so that, in this new equation, the coefficient of $\hat{x}\hat{y}$ is 0.

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$

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&\quad + C(\hat{x}\sin(\theta) + \hat{y}\cos(\theta))(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) \\
&\quad + D(\hat{x}\cos(\theta) - \hat{y}\sin(\theta)) + E(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) + F
\end{aligned}$$

Coefficient of $\hat{x}\hat{y}$:

$$\begin{aligned}
&-A\cos(\theta)\sin(\theta) - A\sin(\theta)\cos(\theta) \\
&\qquad +B\cos(\theta)\cos(\theta) - B\sin(\theta)\sin(\theta) \\
&\qquad +C\sin(\theta)\cos(\theta) + C\cos(\theta)\sin(\theta) \\
&= B(\cos^2(\theta) - \sin^2(\theta)) + (C - A)(2\cos(\theta)\sin(\theta)) \\
&= B\cos(2\theta) + (C - A)\sin(2\theta).
\end{aligned}$$

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta), \quad y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$$

$$\begin{aligned} 0 &= Ax^2 + Bxy + Cy^2 + Dx + Ey + F \\ &= A(\hat{x}\cos(\theta) - \hat{y}\sin(\theta))(\hat{x}\cos(\theta) - \hat{y}\sin(\theta)) \\ &\quad + B(\hat{x}\cos(\theta) - \hat{y}\sin(\theta))(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) \\ &\quad + C(\hat{x}\sin(\theta) + \hat{y}\cos(\theta))(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) \\ &\quad + D(\hat{x}\cos(\theta) - \hat{y}\sin(\theta)) + E(\hat{x}\sin(\theta) + \hat{y}\cos(\theta)) + F \end{aligned}$$

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So we want θ such that

$$B\cos(2\theta) + (C - A)\sin(2\theta) = 0$$

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$$B\cos(2\theta) + (C - A)\sin(2\theta) = 0, \text{ i.e. } \boxed{\tan(2\theta) = B/(A - C)}.$$

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Rotating by θ eliminates the xy terms when $\tan(2\theta) = B/(A - C)$.

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Rotating by θ eliminates the xy terms when
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What θ do you want to rotate by to graph each of the following conic sections?

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What θ do you want to rotate by to graph each of the following conic sections?
Example: $8x^2 - 4xy + 4y^2 + 1 = 0$.

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Example: $8x^2 - 4xy + 4y^2 + 1 = 0$.

Solution: $A = 8$, $B = -4$, $C = 4$

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You try:

1. $x^2 - xy + y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta),\ y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta),\ \tan(2\theta) = B/(A - C).$$

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$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)) \quad \sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

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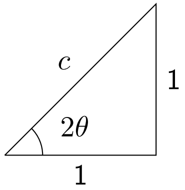

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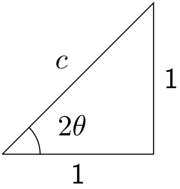

$$c = \sqrt{1^2 + 1^2} = \sqrt{2}$$
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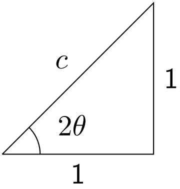

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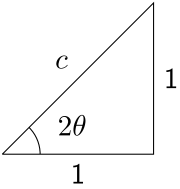

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So

$$x = \hat{x}\sqrt{\frac{1}{2}(1 + \frac{\sqrt{2}}{2})} - \hat{y}\sqrt{\frac{1}{2}(1 - \frac{\sqrt{2}}{2})}, \quad y = \hat{x}\sqrt{\frac{1}{2}(1 - \frac{\sqrt{2}}{2})} + \hat{y}\sqrt{\frac{1}{2}(1 + \frac{\sqrt{2}}{2})}$$

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

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You try: Find the change of coordinates that will eliminate the xy term in each of the following conic sections.

1. $x^2 - xy - y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

Identifying the conic section

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

In the case we did last time, where $B = 0$, we identified the cases

shape	condition if $B = 0$:
parabola	$A = 0$ or $C = 0$
ellipse	A and C same signs
hyperbola	A and C different signs

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parabola	$A = 0$ or $C = 0$	$AC = 0$
ellipse	A and C same signs	$AC > 0$
hyperbola	A and C different signs	$AC < 0$

Just like we did for calculating the coefficient of $\hat{x}\hat{y}$ after changing coordinates, we could calculate all the other coefficients after a change in coordinates

$$0 = \hat{A}(\hat{x})^2 + B\hat{x}\hat{y} + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F.$$

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If we did, we could also show that the discriminants before and after the change are the same, i.e.

$$B^2 - 4AC = (\hat{B})^2 - 4\hat{A}\hat{C}.$$

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$$B^2 - 4AC = (\hat{B})^2 - 4\hat{A}\hat{C}.$$

But if we chose θ so that $\hat{B} = 0$ (like we've been doing), then we get

$$B^2 - 4AC = -4\hat{A}\hat{C}.$$

Putting it together: we start with

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

and convert to

$$0 = \hat{A}(\hat{x})^2 + B\hat{x}\hat{y} + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F,$$

using

$$x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta),\ y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta).$$

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We choose θ such that $\hat{B} = 0$, i.e. $\tan(2\theta) = B/(A - C)$.
Then since $B^2 - 4AC = (\hat{B})^2 - 4\hat{A}\hat{C}$, we get

$$-4\hat{A}\hat{C} = B^2 - 4AC.$$

Putting it together: we start with

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Using our table from before, we can classify the conic section according to the following table:

shape	condition
parabola	$\hat{A}\hat{C} = 0$
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Putting it together: we start with

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You try

Identify the kind of conic section given by each of the following equations:

1. $x^2 - xy - y^2 - 2 = 0$
2. $4x^2 - 4xy + 7y^2 - 24 = 0$
3. $2x^2 + 4\sqrt{3}xy + 6y^2 - 9x + 9\sqrt{3}y - 63 = 0$

Graphing $0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$

1. If $B \neq 0$, change coordinates using $x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta)$, $y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$, where $\tan(2\theta) = B/(A - C)$. Tip: Use a triangle and the half-angle formulas to calculate $\cos(\theta)$ and $\sin(\theta)$.

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   Tip: Use a triangle and the half-angle formulas to calculate $\cos(\theta)$ and $\sin(\theta)$.
2. With your new equation
   $$0 = \hat{A}(\hat{x})^2 + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F,$$
   get into one of the canonical forms and calculate all the important features of the graph.
   (If $B = 0$, then $\hat{A} = A$, $\hat{C} = C$, etc..)

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   (If $B = 0$, then $\hat{A} = A$, $\hat{C} = C$, etc..)
3. Sketch the graph by first sketching the rotated axes and then graphing the ˆ curve on those axes.

Graphing $0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$

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   get into one of the canonical forms and calculate all the important features of the graph.
   (If $B = 0$, then $\hat{A} = A$, $\hat{C} = C$, etc..)
3. Sketch the graph by first sketching the rotated axes and then graphing the ˆ curve on those axes.
4. If needed, convert features of the graph (like vertices, foci, asymptotes, etc.) back into x, y-coordinates using
   $$\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta).$$

Graphing $0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$

1. If $B \neq 0$, change coordinates using $x = \hat{x}\cos(\theta) - \hat{y}\sin(\theta)$, $y = \hat{x}\sin(\theta) + \hat{y}\cos(\theta)$, where $\tan(2\theta) = B/(A - C)$.
   Tip: Use a triangle and the half-angle formulas to calculate $\cos(\theta)$ and $\sin(\theta)$.
2. With your new equation
   $$0 = \hat{A}(\hat{x})^2 + C(\hat{y})^2 + D\hat{x} + E\hat{y} + F,$$
   get into one of the canonical forms and calculate all the important features of the graph.
   (If $B = 0$, then $\hat{A} = A$, $\hat{C} = C$, etc..)
3. Sketch the graph by first sketching the rotated axes and then graphing the $\hat{}$ curve on those axes.
4. If needed, convert features of the graph (like vertices, foci, asymptotes, etc.) back into x, y-coordinates using
   $$\hat{x} = x\cos(\theta) + y\sin(\theta) \quad \text{and} \quad \hat{y} = -x\sin(\theta) + y\cos(\theta).$$

You try: Sketch a graph of $4x^2 - 4xy + 7y^2 - 24 = 0$.