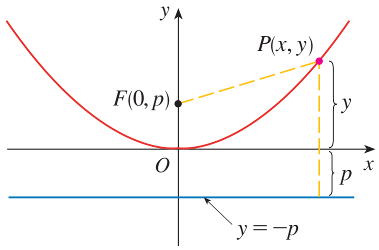
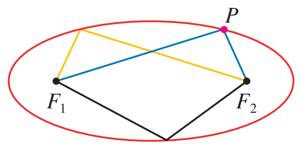
Review of conic sections

Conic sections are graphs of the form

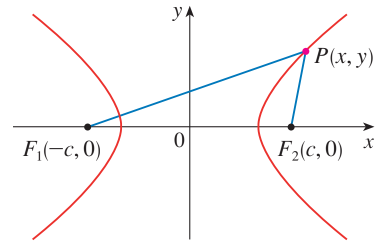
parabolas



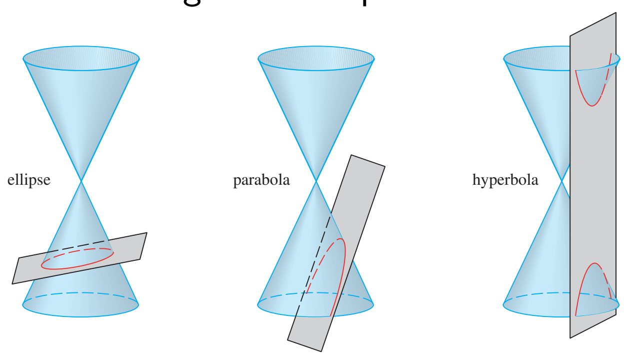
ellipses



hyperbolas



They're called conic sections because they can be found by taking a doubled cone and slicing it with a plane:



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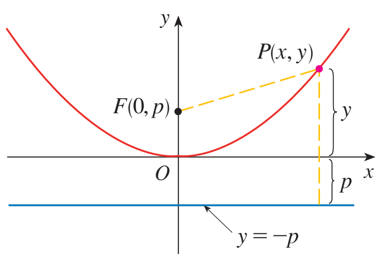

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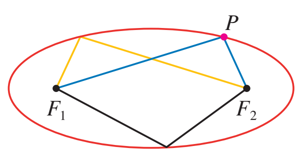
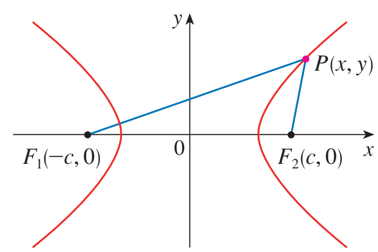


hyperbolas

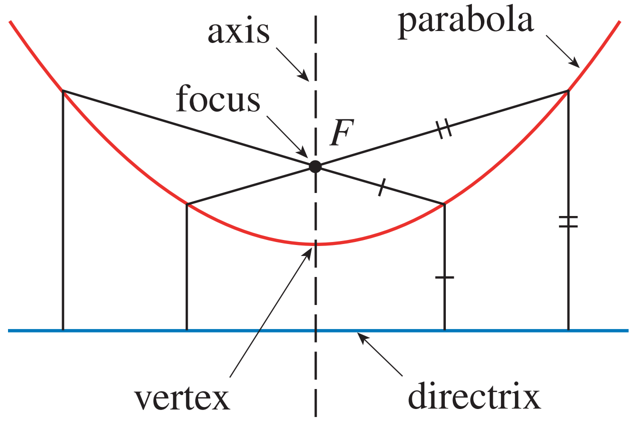


All conic sections satisfy an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for some constants A, B, C, D, E, F.

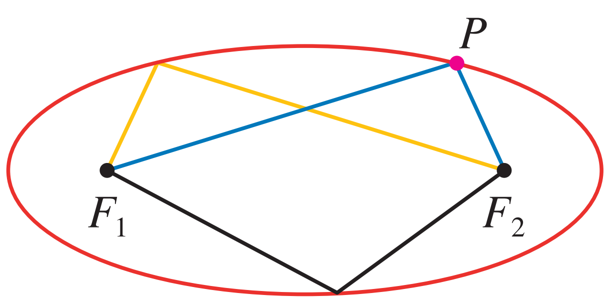
Parabolas



Parabolas have a focus F (a point) and a directrix (a line), and are defined as all the points that are equidistant from the focus and the directrix.

They also have a vertex (the point on the curve closest to the directrix) and an axis (the line perpendicular to the directrix through the vertex and the focus).

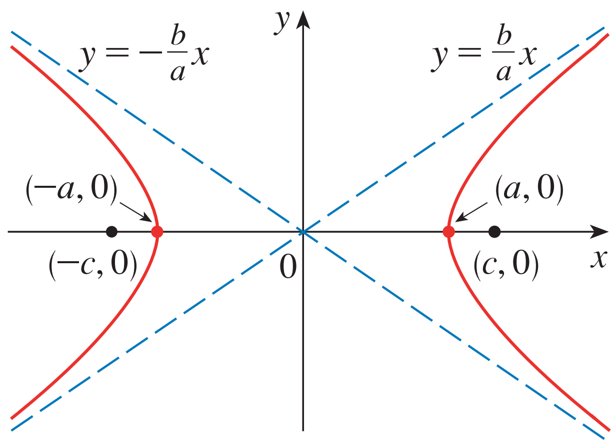
Ellipses



Ellipses have two foci F_1 and F_2 (both points), and are defined as all the points (x, y) such that the distance from F_1 to (x, y) plus the distance from (x, y) to F_2 is fixed.

They also have a major axis (the line through the foci) and vertices (the points on the curve intersecting the major axis).

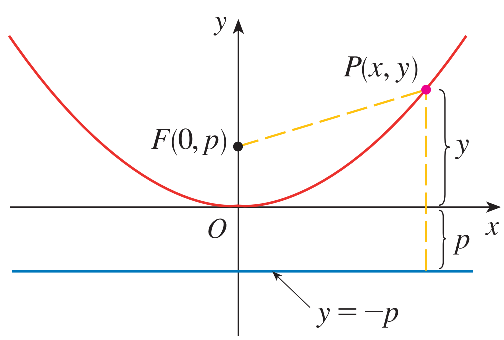
Hyperbolas



Hyperbolas also have two foci F_1 and F_2 (both points), and are defined as all the points (x, y) such that the difference between the distances from F_1 to (x, y) and from (x, y) to F_2 is fixed.

They also have vertices (one point on each piece which are closest to each other) and asymptotes (lines which the curves approach at infinity).

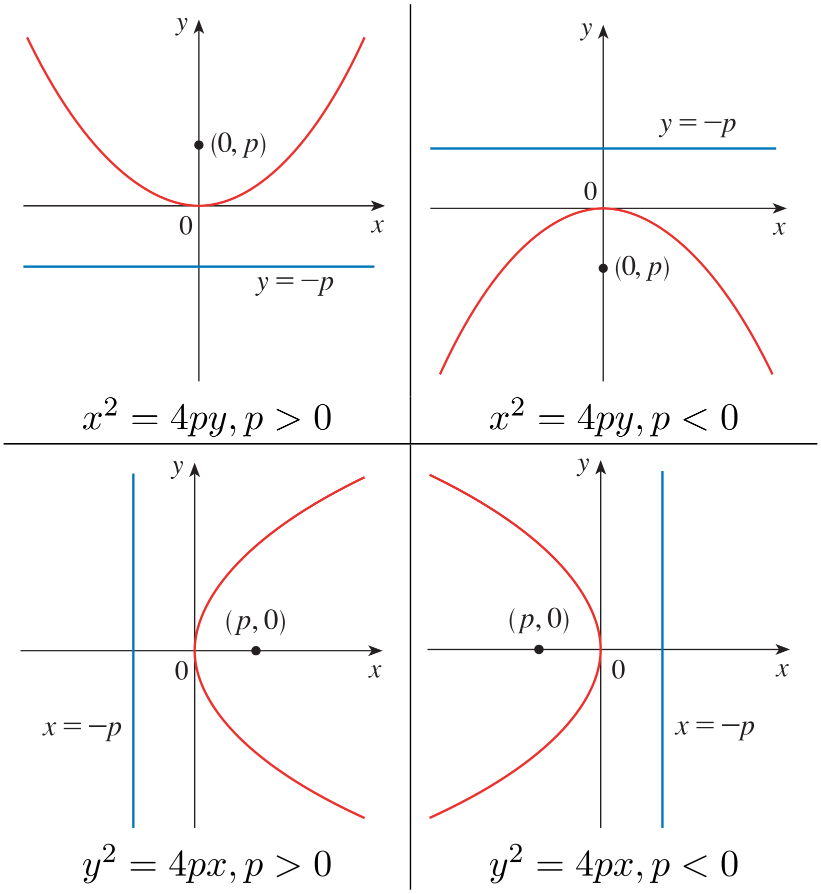
Calculating equations for conic sections: Parabolas



For a parabola with vertex at the origin and directrix $y = -p$ parallel to the x-axis, the definition "all the points $P(x, y)$ that are equidistant from the focus and the directrix" says first that the focus had better be at $(0, p)$ (since the origin is on the curve), and then that

$$y + p = |PF| = \sqrt{(x-0)^2 + (y-p)}.$$

Solving for x^2, this gives $\boxed{x^2 = 4py}$. Similarly, if the parabola had vertex at the origin and directrix $x = -p$, then the focus is at $(p, 0)$ and the equation for the parabola is $\boxed{y^2 = 4px}$.



Examples of parabola problems

1. What conic section is the curve $10y = x^2$? What are all relevant points and lines?

Ans. This is a parabola with vertex $(0, 0)$, focus $(0, p)$ and directrix $y = -p$ where $y = 4px^2$ (since x is the squared variable). Thus $p = 10/4 = 5/2$.

2. What conic section is the curve $10(y - 1) = (x + 2)^2$?

Ans. Shift the coordinates! Let $\hat{y} = y - 1$ (so $y = \hat{y} + 1$) and $\hat{x} = x + 2$ (so $x = \hat{x} - 2$). Then $\hat{y} = 10(\hat{x})^2$ is the parabola in part 1, and we just have to shift all of our coordinates back.

$$\text{vertex: } (\hat{x}, \hat{y}) = (0, 0), \text{ so } (x, y) = (0 - 2, 0 + 1) = (-2, 1)$$

$$\text{focus: } (\hat{x}, \hat{y}) = (0, 5/2), \text{ so } (x, y) = (0 - 2, 5/2 + 1) = (-2, 7/2)$$

$$\text{directrix: } \hat{y} = -5/2, \text{ so } y = -5/2 + 1 = -3/2$$

Examples of parabola problems

2. What conic section is the curve $10(y-1) = (x+2)^2$?
Ans.

$$\begin{aligned}\text{vertex: } & (\hat{x}, \hat{y}) = (0,0), \text{ so } (x,y) = (0-2, 0+1) = (-2,1) \\ \text{focus: } & (\hat{x}, \hat{y}) = (0,5/2), \text{ so } (x,y) = (0-2, 5/2+1) = (-2,7/2) \\ \text{directrix: } & \hat{y} = -5/2, \text{ so } y = -5/2+1 = -3/2\end{aligned}$$

3. What conic section is the curve $10y - x^2 - 4x - 14 = 0$?
Ans. Try to put in a form we recognize! Deal with the x and y stuff separately and complete whatever squares appear.

$$\begin{aligned}-x^2 - 4x &= -(x^2+4x) = -(x^2+4x+4-4) \\ &= -((x+2)^2-4) = \boxed{-(x+2)^2+4};\end{aligned}$$

So

$$\begin{aligned}0 &= 10y + \boxed{-x^2-4x} - 14 \\ &= 10y + \boxed{-(x+2)^2+4} - 14 = 10(y-1) - (x+2)^2.\end{aligned}$$

Same as part 2!

Doing parabola problems

Look for x^2 or y^2, but not both!
Get into the proper form!

$$(\hat{x})^2 = 4p\hat{y} \text{ or } (\hat{y})^2 = 4p\hat{x}.$$

If you're given something like $10y - x^2 - 4x - 14 = 0$, separate the x stuff and the y stuff and complete squares, and deal with shifting coordinates if necessary.

1–8 ■ Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x = 2y^2$

2. $4y + x^2 = 0$

3. $4x^2 = -y$

4. $y^2 = 12x$

5. $(x + 2)^2 = 8(y - 3)$

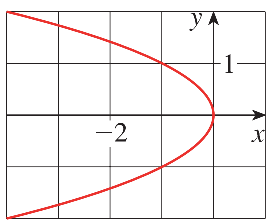
6. $x - 1 = (y + 5)^2$

7. $y^2 + 2y + 12x + 25 = 0$

8. $y + 12x - 2x^2 = 16$

9–10 ■ Find an equation of the parabola. Then find the focus and directrix.

9.



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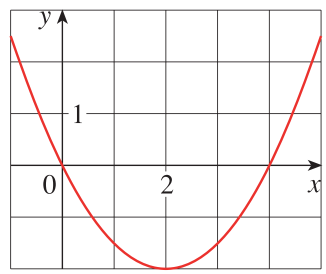
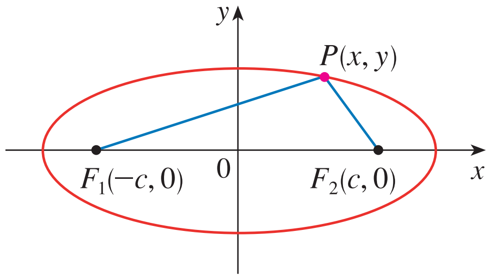
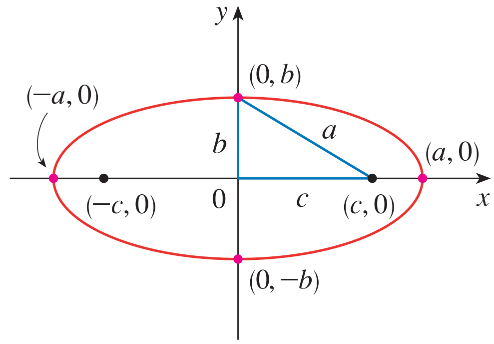


Calculating equations for conic sections: Ellipses





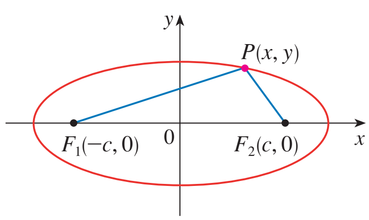
For an ellipse with the foci on the x-axis at the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ and the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then for any point $P = P(x, y)$ on the curve

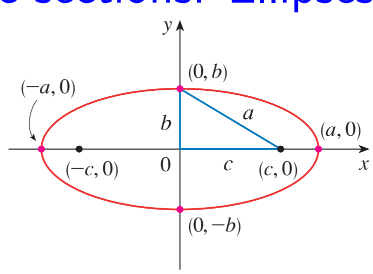
$$2a = |PF_1| + |PF_2| = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}.$$

Since $a > c$, let b be defined by $b^2 = a^2 - c^2$. Manipulating the equation above gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Calculating equations for conic sections: Ellipses





For an ellipse with the foci on the x-axis at the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ and the sum of the distances from a point on the ellipse to the foci be $2a > 0$,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \text{ where } b^2 = a^2 - c^2.$$

The major axis is the x-axis and the vertices are $(\pm a, 0)$. Similarly, if the foci are the points $F_1 = (0, -c)$ and $F_2 = (0, c)$ and the sum of the distances from a point on the ellipse to the foci be $2a > 0$,

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1, \text{ where } b^2 = a^2 - c^2.$$

The major axis is the y-axis and the vertices are $(0, \pm a)$.

Doing ellipse problems

Look for both x^2 and y^2, with the same sign.
Get into the proper form!

$$\left(\frac{\hat{x}}{a}\right)^2 + \left(\frac{\hat{y}}{b}\right)^2 = 1.$$

For example, if you're given

$$9x^2 + 16y^2 = 144,$$

divide both sides by 144 first and factor into squares:

$$1 = \frac{9}{144}x^2 + \frac{16}{144}y^2 = \frac{x^2}{144/9} + \frac{y^2}{144/16} = \frac{x^2}{16} + \frac{y^2}{9} = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2.$$

If you're given something like

$$y^2 + 2x^2 - 2y + 8x = 0$$

separate the x stuff and the y stuff and complete squares, and deal with shifting coordinates if necessary.

11–16 ■ Find the vertices and foci of the ellipse and sketch its graph.

11. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

12. $\frac{x^2}{64} + \frac{y^2}{100} = 1$

13. $4x^2 + y^2 = 16$

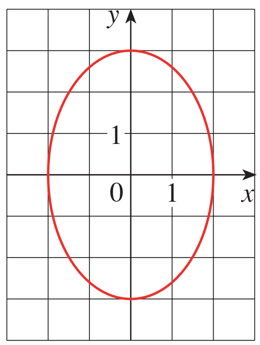
14. $4x^2 + 25y^2 = 25$

15. $9x^2 - 18x + 4y^2 = 27$

16. $x^2 + 2y^2 - 6x + 4y + 7 = 0$

17–18 ■ Find an equation of the ellipse. Then find its foci.

17.



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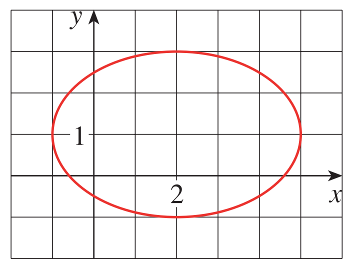
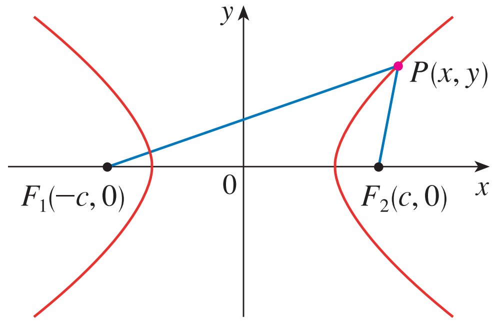


Calculating equations for conic sections: Hyperbolas



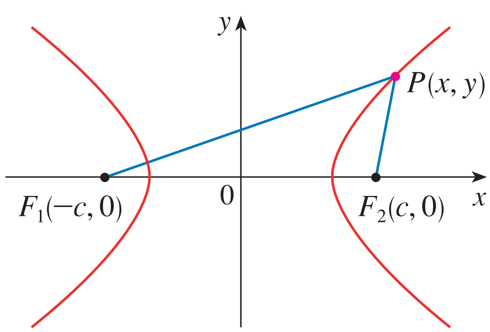
For a hyperbola with the foci on the x-axis at the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ and the difference of the distances from a point on the ellipse to the foci be $\pm 2a$ ($a > 0$). Then for any point $P = P(x, y)$ on the curve

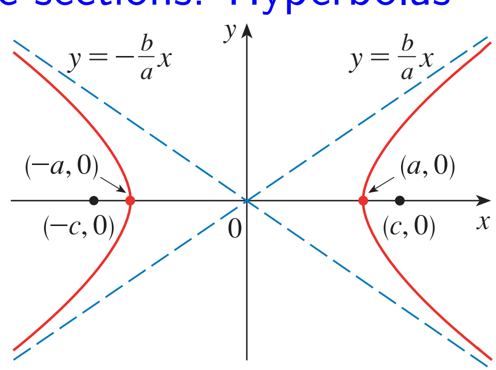
$$\pm 2a = |PF_1| - |PF_2| = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}.$$

Let b be defined by $c^2 = b^2 + a^2$. Manipulating the equation above gives

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

Calculating equations for conic sections: Hyperbolas



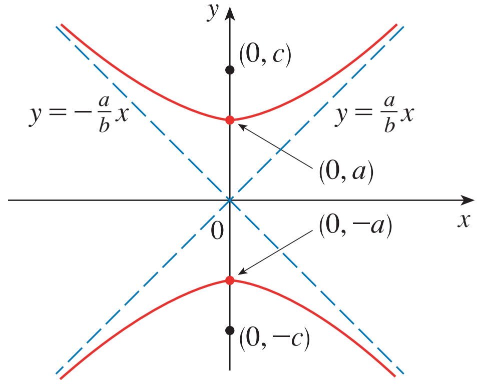


For a hyperbola with the foci on the x-axis at the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ and the difference of the distances from a point on the ellipse to the foci be $\pm 2a$ ($a > 0$),

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \text{ where } c^2 = b^2 + a^2.$$

Note that the x-intercepts are $\pm a$ (set $y = 0$ and solve). There is no y-intercept ($x = 0$ has no solutions). But as $x \to \pm\infty$, $y/x \to \pm b/a$. So the vertices are $(\pm a, 0)$ and the asymptotes are $y = \pm(b/a)x$.

Calculating equations for conic sections: Hyperbolas



Similarly, for a hyperbola with the foci on the y-axis at the points $F_1 = (0, -c)$ and $F_2 = (0, c)$ and the difference of the distances from a point on the ellipse to the foci be $\pm 2a$ $(a > 0)$,

$$\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1, \text{ where } c^2 = b^2 + a^2.$$

The y-intercepts are $\pm a$ (set $x = 0$ and solve). There is no x-intercept ($y = 0$ has no solutions). And as $x \to \pm\infty$, $y/x \to \pm a/b$. So the vertices are $(0, \pm a)$ and the asymptotes are $y = \pm(a/b)x$. (Switch all x's for y's.)

Doing hyperbola problems

Look for both x^2 and y^2, with different signs.
Get into the proper form!

$$\left(\frac{\hat{x}}{a}\right)^2 - \left(\frac{\hat{y}}{b}\right)^2 = 1 \text{ or } \left(\frac{\hat{y}}{a}\right)^2 - \left(\frac{\hat{x}}{b}\right)^2 = 1$$

For example, if you're given

$$9x^2 - 16y^2 = 144,$$

divide both sides by 144 first and factor into squares:

$$1 = \frac{9}{144}x^2 - \frac{16}{144}y^2 = \frac{x^2}{144/9} - \frac{y^2}{144/16} = \frac{x^2}{16} - \frac{y^2}{9} = \left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2.$$

If you're given something like

$$y^2 - 2x^2 - 2y - 8x = 0$$

separate the x stuff and the y stuff and complete squares, and deal with shifting coordinates if necessary.

19–20 ■ Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$

20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1$

21. $y^2 - x^2 = 4$

22. $9x^2 - 4y^2 = 36$

23. $2y^2 - 3x^2 - 4y + 12x + 8 = 0$

24. $16x^2 - 9y^2 + 64x - 90y = 305$

25–30 ■ Identify the type of conic section whose equation is given and find the vertices and foci.

25. $x^2 = y + 1$

26. $x^2 = y^2 + 1$

27. $x^2 = 4y - 2y^2$

28. $y^2 - 8y = 6x - 16$

29. $y^2 + 2y = 4x^2 + 3$

30. $4x^2 + 4x + y^2 = 0$

EXERCISES

A Click here for answers. S Click here for solutions.

1–8 ■ Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x = 2y^2$

2. $4y + x^2 = 0$

3. $4x^2 = -y$

4. $y^2 = 12x$

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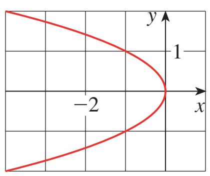
6. $x - 1 = (y + 5)^2$

7. $y^2 + 2y + 12x + 25 = 0$

8. $y + 12x - 2x^2 = 16$

9–10 ■ Find an equation of the parabola. Then find the focus and directrix.

9.



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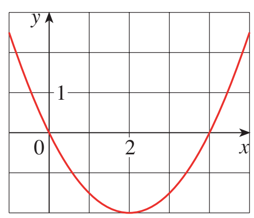

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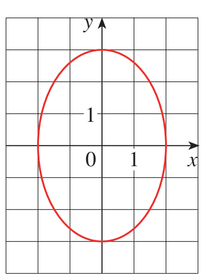

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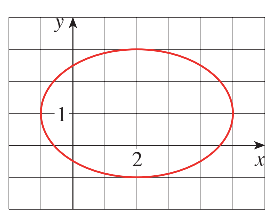

19–20 ■ Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$

20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1$

21. $y^2 - x^2 = 4$

22. $9x^2 - 4y^2 = 36$

23. $2y^2 - 3x^2 - 4y + 12x + 8 = 0$

24. $16x^2 - 9y^2 + 64x - 90y = 305$

25–30 ■ Identify the type of conic section whose equation is given and find the vertices and foci.

25. $x^2 = y + 1$

26. $x^2 = y^2 + 1$

27. $x^2 = 4y - 2y^2$

28. $y^2 - 8y = 6x - 16$

29. $y^2 + 2y = 4x^2 + 3$

30. $4x^2 + 4x + y^2 = 0$

31–48 ■ Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex $(0, 0)$, focus $(0, -2)$

32. Parabola, vertex $(1, 0)$, directrix $x = -5$

33. Parabola, focus $(-4, 0)$, directrix $x = 2$

34. Parabola, focus $(3, 6)$, vertex $(3, 2)$

35. Parabola, vertex $(0, 0)$, axis the x-axis, passing through $(1, -4)$

36. Parabola, vertical axis, passing through $(-2, 3)$, $(0, 3)$, and $(1, 9)$

37. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$

38. Ellipse, foci $(0, \pm 5)$, vertices $(0, \pm 13)$

39. Ellipse, foci $(0, 2)$, $(0, 6)$ vertices $(0, 0)$, $(0, 8)$

40. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$

41. Ellipse, center $(2, 2)$, focus $(0, 2)$, vertex $(5, 2)$

42. Ellipse, foci $(\pm 2, 0)$, passing through $(2, 1)$

43. Hyperbola, foci $(0, \pm 3)$, vertices $(0, \pm 1)$

44. Hyperbola, foci $(\pm 6, 0)$, vertices $(\pm 4, 0)$

45. Hyperbola, foci $(1, 3)$ and $(7, 3)$, vertices $(2, 3)$ and $(6, 3)$

46. Hyperbola, foci $(2, -2)$ and $(2, 8)$, vertices $(2, 0)$ and $(2, 6)$

47. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$

48. Hyperbola, foci $(2, 2)$ and $(6, 2)$, asymptotes $y = x - 2$ and $y = 6 - x$