Warm up

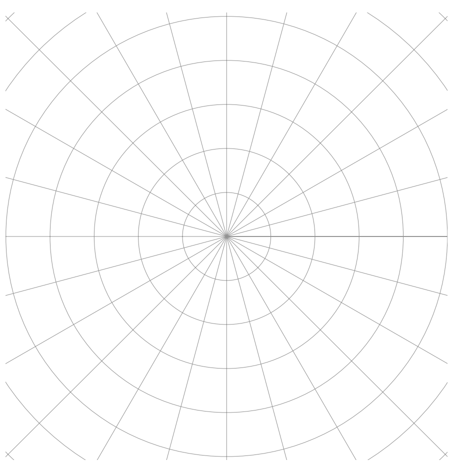
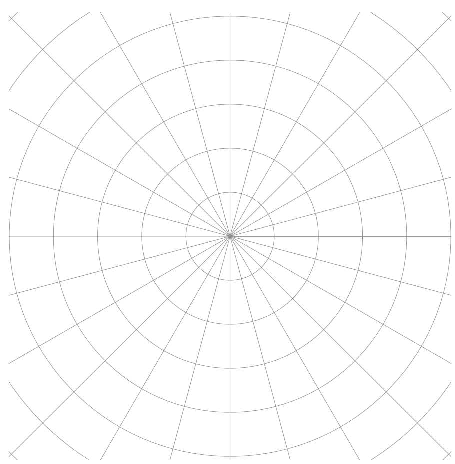
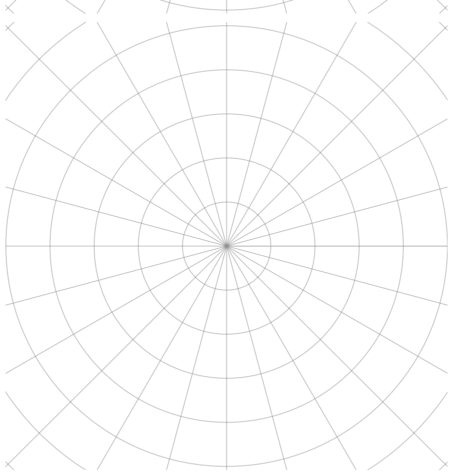
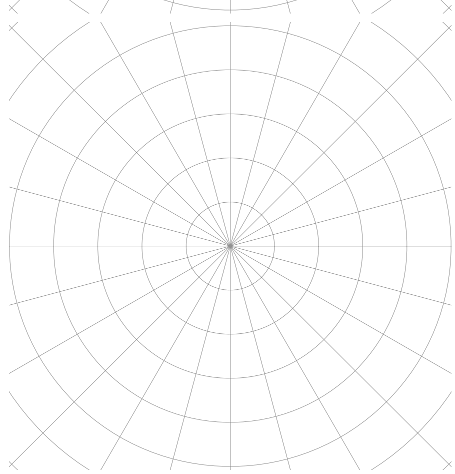
Recall from last time, given a polar curve $r = r(\theta)$,

$$\boxed{\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{\frac{d}{d\theta}(r(\theta)\sin(\theta))}{\frac{d}{d\theta}(r(\theta)\cos(\theta))} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}.}$$

1. Last time, we calculated that for the cardioid $r(\theta) = 1 + \sin(\theta)$,

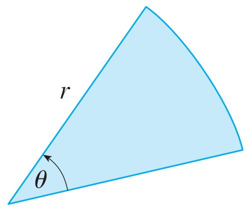
   $$\frac{dy}{d\theta} = \cos(\theta)(1 + 2\sin(\theta)) \qquad \frac{dx}{d\theta} = (1 + \sin(\theta))(1 - 2\sin(\theta)).$$

   For what θ are the tangent lines to this cardioid horizontal? vertical?
2. Calculate dy/dx for the following polar curves. What are the slopes at which each curve crosses the x and y axes?
   (a) $r(\theta) = 2$
   (b) $r(\theta) = \theta$
   (c) $r(\theta) = \cos(2\theta)$

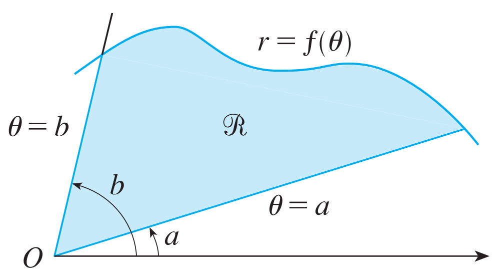
Area

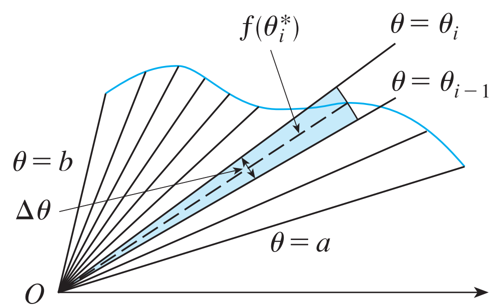
Recall that the area of a wedge of a circle with angle θ is given by



$$A = \pi r^2(\theta/2\pi) = \tfrac{1}{2}\theta r^2.$$

Let $r = f(\theta)$ be a positive continuous function for θ over $[a, b]$, and let R be the region bounded by r and the rays $\theta = a$ and $\theta = b$.

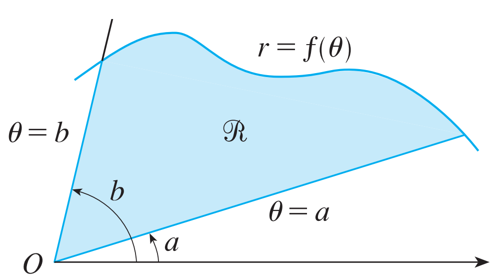


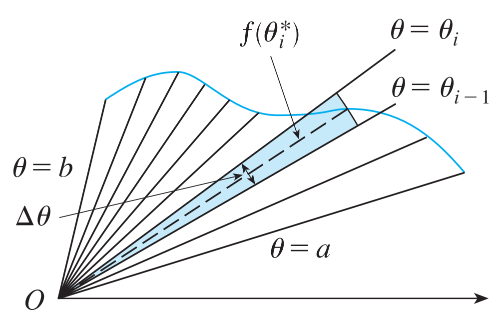


We can approximate the area of R by slicing it into wedges!

Area

Let $r = f(\theta)$ be a positive continuous function for θ over $[a, b]$, and let R be the region bounded by r and the rays $\theta = a$ and $\theta = b$.





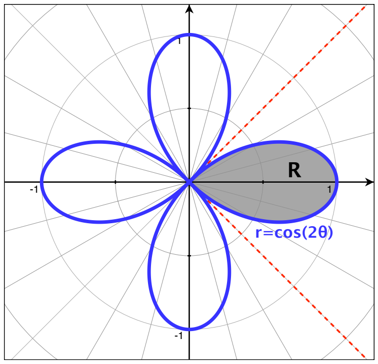
We can approximate the area of R by slicing it into wedges! The area of each wedge is approximately

$$\Delta A \approx \tfrac{1}{2} r_i^2 \Delta\theta = \tfrac{1}{2} f(\theta_i^*)^2 \Delta\theta. \qquad \text{So } A = \int_a^b \tfrac{1}{2} r^2 \, d\theta = \int_a^b \tfrac{1}{2} f(\theta)^2 \, d\theta.$$

Example

$$\text{For } r = r(\theta) \text{ positive, } \boxed{A = \int_a^b \tfrac{1}{2} r^2 \; d\theta}$$

Find the area enclosed by one loop of the curve $r = \cos(2\theta)$.



$$A = \int_a^b \tfrac{1}{2} r^2 \; d\theta$$

$$= \int_a^b \tfrac{1}{2} (\cos(2\theta))^2 \; d\theta$$

One choice of a and b: $a = -\pi/4$, $b = -\pi/4$.

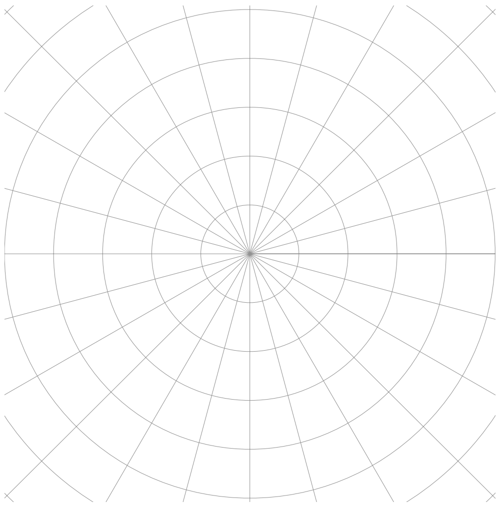
$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} \cos^2(2\theta) \; d\theta = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}(1 + \cos(4\theta)) \; d\theta$$

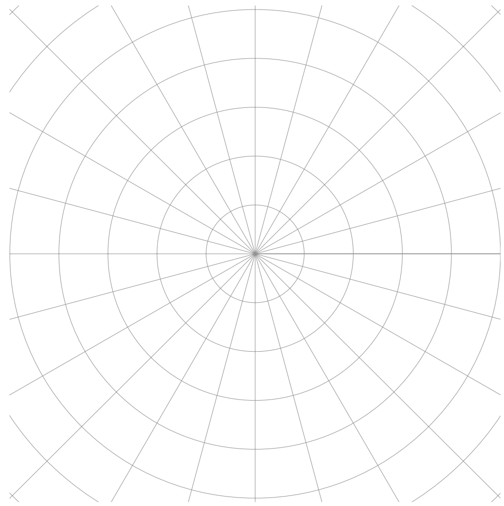
$$= \tfrac{1}{4}(\theta + \tfrac{1}{4}\sin(4\theta))\Big|_{-\pi/4}^{\pi/4} = \tfrac{1}{4}(\pi/4 + 0) - \tfrac{1}{4}(-\pi/4 + 0) = 2\pi/16 = \pi/8.$$

You try

For $r = r(\theta)$ positive, $\boxed{A = \int_a^b \frac{1}{2} r^2 \; d\theta}$

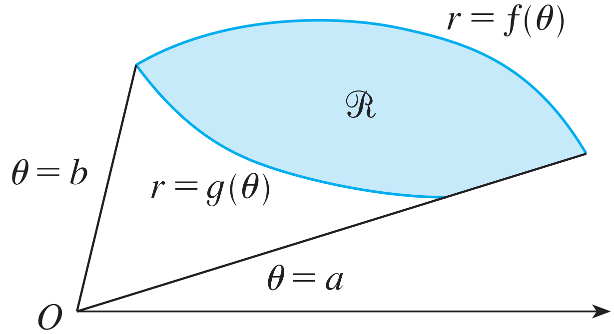
1. Calculate the area enclosed by the cardioid $r = 1 + \sin(x)$.
2. Graph the curve $r = \sin(2\theta)$ and calculate the area enclosed by this curve. (Careful! The function $\sin(2\theta)$ isn't always positive, and we want *area*! Look at the picture and decide how to break up the problem.)





Area beween curves

Let R be the region bounded between curves $r = f(\theta)$ and $r = g(\theta)$ (where $0 \leq g(\theta) \leq f(\theta)$) and rays $\theta = a$ and $\theta = b$.



The area of R is given by

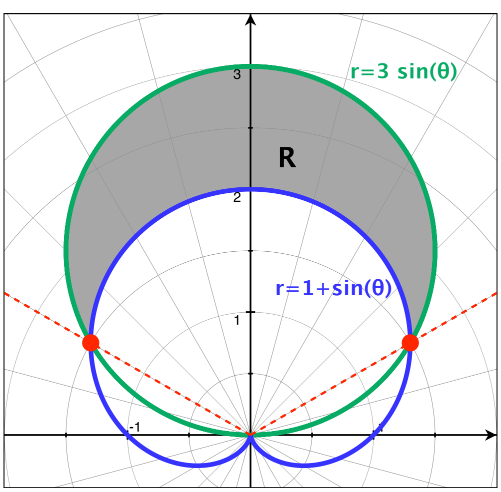
$$A = \int_a^b \tfrac{1}{2} r_{\text{out}}^2 \, d\theta - \int_a^b \tfrac{1}{2} r_{\text{in}}^2 \, d\theta = \int_a^b \tfrac{1}{2}(r_{\text{out}}^2 - r_{\text{in}}^2) \, d\theta$$
$$= \int_a^b \tfrac{1}{2}(f^2(\theta) - g^2(\theta)) \, d\theta$$

Area beween curves

For $0 \leq r_{\text{in}} \leq r_{\text{out}}$, $\boxed{A = \int_a^b \tfrac{1}{2}((r_{\text{out}}^2 - r_{\text{in}}^2)) \, d\theta}$

Example: Find the area of the region that lies inside the circle $r = 3\sin(\theta)$ and outside the cardioid $r = 1 + \sin(\theta)$.

Answer: First, graph the functions.



Next, identify the region and the bounds.

Intersection points:

$3\sin(\theta) = 1+\sin(\theta)$, so $\sin(\theta) = \frac{1}{2}$,

so $\boxed{\theta = \pi/6,\ 5\pi/6}$.

Setting up the integral:

$r_{\text{out}} = 3\sin(\theta), \quad r_{\text{in}} = 1 + \sin(\theta)$

(Check! Are both r's positive over the interval? If not, do we have to fix anything?) ✓

So $A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}((3\sin(\theta))^2 - (1 + \sin(\theta))^2) \, d\theta$

You try:

For $r = r(\theta)$ positive, $\boxed{A = \int_a^b \frac{1}{2} r^2 \, d\theta}$

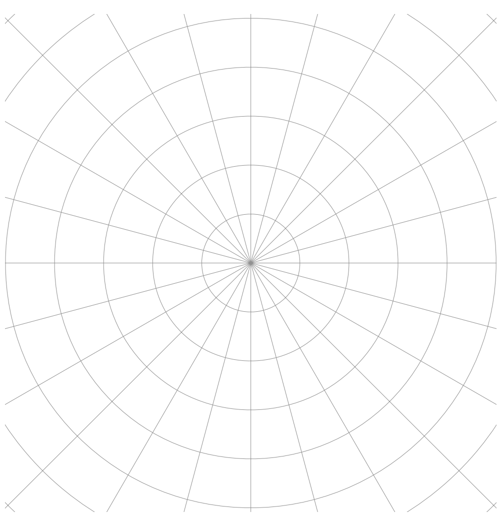
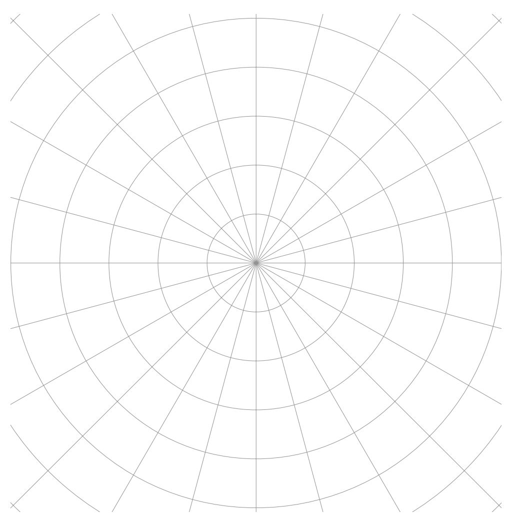
For $0 \leq r_{\text{in}} \leq r_{\text{out}}$, $\boxed{A = \int_a^b \frac{1}{2}((r_{\text{out}}^2 - r_{\text{in}}^2)) \, d\theta}$

1. We showed that the area of the region that lies inside the circle $r = 3\sin(\theta)$ and outside the cardioid $r = 1 + \sin(\theta)$ is given by the integral

$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}((3\sin(\theta))^2 - (1 + \sin(\theta))^2) \, d\theta.$$

   Check that this integral evaluates to π.

2. For the following regions, (i) graph the functions and identify the region, (ii) decide how to break your integral up using the above area formulas, (ii) calculate the area of the region.
   (a) Let R be the region inside $r = (1/2)\sin(2\theta)$ and outside $r = 1/4$.
   (b) Let R be the region enclosed by both $r = 1 + \sin(\theta)$ and $r = 1 - \sin(\theta)$.

Arc length (again!!)

Again! We start with

$$\ell = \int d\ell, \qquad \text{where } d\ell = \sqrt{dx^2 + dy^2}.$$

When we had $y = f(x)$, we multiplied by dx/dx to get

$$d\ell = \sqrt{1 + (dy/dx)^2}\ dx.$$

When we had $x = f(y)$, we multiplied by dy/dy to get

$$d\ell = \sqrt{(dx/dy)^2 + 1}\ dy.$$

When we had $x(t)$ and $y(t)$, we multiplied by dt/dt to get

$$d\ell = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\ dt.$$

Now we have

$$x(\theta) = r(\theta)\cos(\theta) \quad \text{and} \quad y(\theta) = r(\theta)\sin(\theta),$$

so multiply by $d\theta/d\theta$ to get

$$\boxed{d\ell = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta}$$

(exactly the same as the t case above)

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$$\boxed{d\ell = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta}$$

(exactly the same as the t case above), now where

$$\frac{dx}{d\theta} = \frac{d}{d\theta} r(\theta)\cos(\theta) = \boxed{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}, \text{ and}$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} r(\theta)\sin(\theta) = \boxed{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}.$$

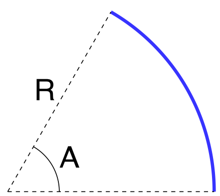
Arc length of parametric curves

$$\boxed{\ell = \int_{\theta=a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}, \text{ where } \begin{array}{l} x(\theta) = r(\theta)\cos(\theta), \\ y(\theta) = r(\theta)\sin(\theta). \end{array}$$

Example: Again! Let's calculate the arc length of an arc of angle A of a circle of radius R. The circle of radius R is given by the polar curve $r(\theta) = R$. So

$$x(\theta) = R\cos(\theta) \quad y(\theta) = R\sin(\theta),$$

and an arc of angle A is the curve traced from $\theta = 0$ to $\theta = A$:



$$dx/d\theta = -R\sin(\theta)$$
$$dy/d\theta = R\cos(\theta)$$
$$d\ell = \sqrt{(-R\sin(\theta))^2 + (R\cos(\theta))^2}\, d\theta = \sqrt{R^2}\, dt$$

So $\ell = \int_{\theta=0}^{A} d\ell = \int_0^A R\, d\theta = R\theta\big|_{\theta=0}^{A} = RA - 0 = \boxed{RA}$.

You try

$$\boxed{\ell = \int_{\theta=a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}, \text{ where } \begin{array}{l} x(\theta) = r(\theta)\cos(\theta), \\ y(\theta) = r(\theta)\sin(\theta). \end{array}$$

1. Set up and simplify an integral that gives the length of the curve $r = \theta$ from $\theta = \pi/2$ to $\theta = \pi$.
2. Calculate the length of the curve $r = \cos(\theta)$ from $\theta = 0$ to $\theta = \pi$. (Hint: after doing derivatives, before setting up, recall double angle formulas.)
3. Calculate the length of the curve $r = e^{\theta}$, from $\theta = 1$ to $\theta = 2$.