Warm up

Recall from last time, given a polar curve $r = r(\theta)$,

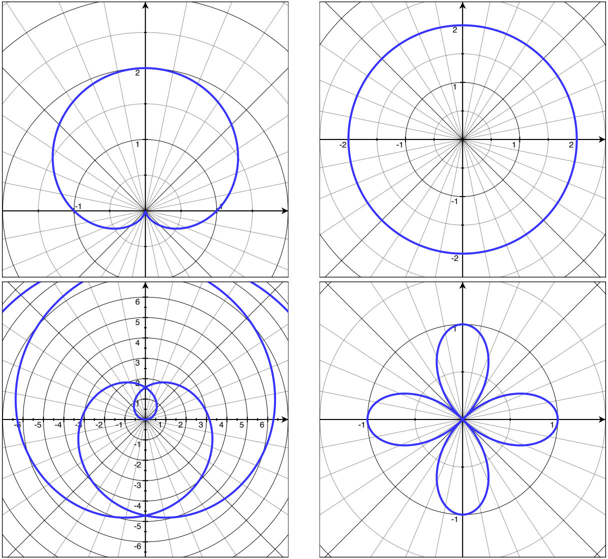
$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\frac{d}{d\theta}(r(\theta)\sin(\theta))}{\frac{d}{d\theta}(r(\theta)\cos(\theta))} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}.$$

1. Last time, we calculated that for the cardioid $r(\theta) = 1 + \sin(\theta)$,

   $$\frac{dy}{d\theta} = \cos(\theta)(1 + 2\sin(\theta)) \qquad \frac{dx}{d\theta} = (1 + \sin(\theta))(1 - 2\sin(\theta)).$$

   For what θ are the tangent lines to this cardioid horizontal? vertical?

2. Graph the following polar curves, and calculate dy/dx. What are the slopes at which each curve crosses the x and y axes?
   (a) $r(\theta) = 2$
   (b) $r(\theta) = \theta$
   (c) $r(\theta) = \cos(2\theta)$



2
1
-1
1
2
1
-2
-1
1
2
-1
-2
6
5
4
3
2
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-6
-5
-4
-3
-2
-1
1
2
3
4
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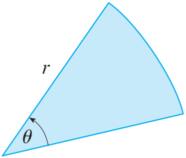
-1
1
-1

Area

Recall that the area of a wedge of a circle with angle θ is given by



$$A = \pi r^2(\theta/2\pi) = \frac{1}{2}\theta r^2.$$

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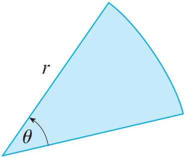
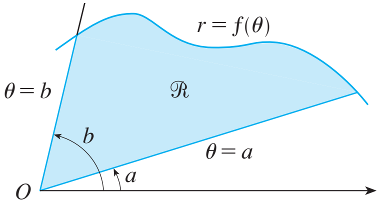


$$A = \pi r^2(\theta/2\pi) = \frac{1}{2}\theta r^2.$$

Let $r = f(\theta)$ be a positive continuous function for θ over $[a, b]$, and let R be the region bounded by r and the rays $\theta = a$ and $\theta = b$.



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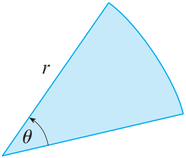

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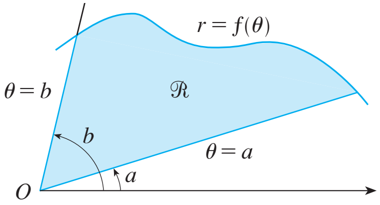

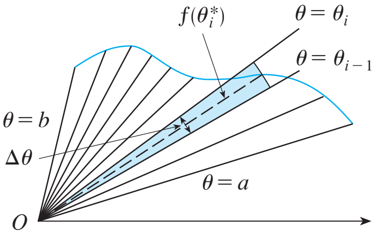

We can approximate the area of R by slicing it into wedges!

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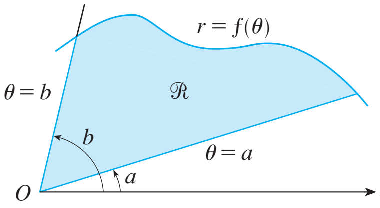

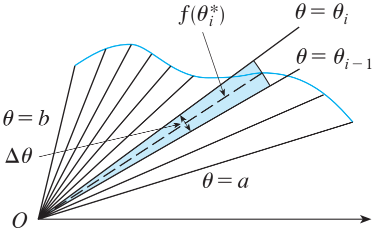

We can approximate the area of R by slicing it into wedges! The area of each wedge is approximately

$$\Delta A \approx \frac{1}{2} r_i^2 \Delta\theta = \frac{1}{2} f(\theta_i^*)^2 \Delta\theta.$$

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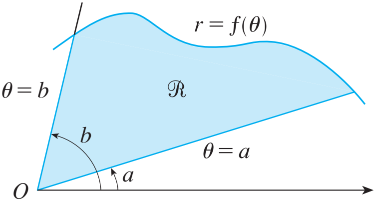
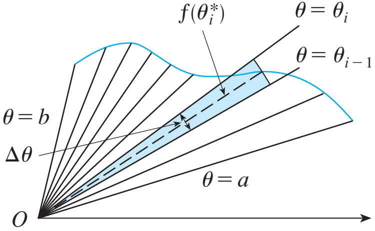




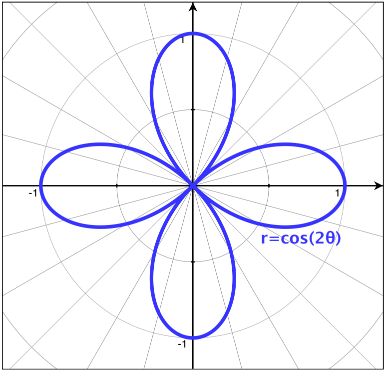
We can approximate the area of R by slicing it into wedges! The area of each wedge is approximately

$\Delta A \approx \frac{1}{2} r_i^2 \Delta\theta = \frac{1}{2} f(\theta_i^*)^2 \Delta\theta.$ So $A = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta.$

Example

For $r = r(\theta)$ positive, $\boxed{A = \int_a^b \frac{1}{2} r^2 \, d\theta}$

Find the area enclosed by one loop of the curve $r = \cos(2\theta)$.



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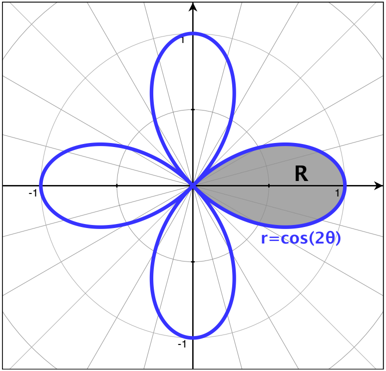

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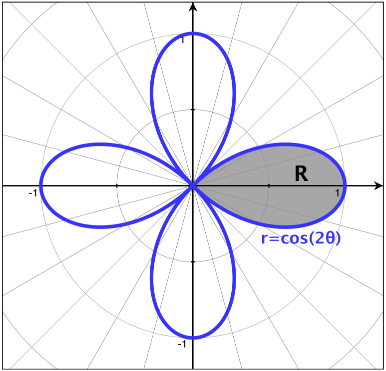

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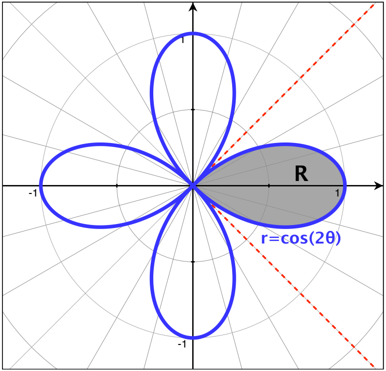

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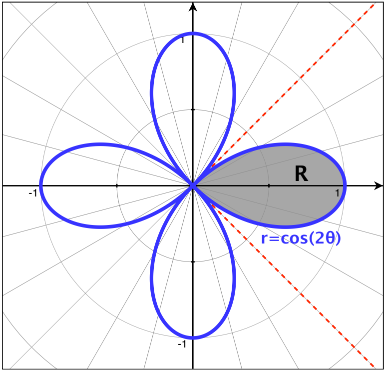

$$
\begin{aligned}
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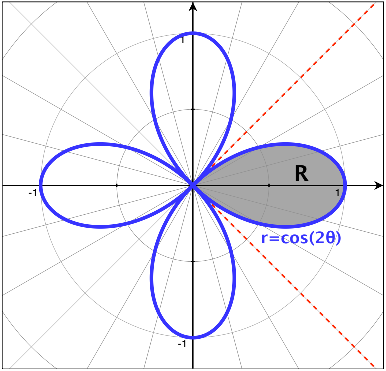

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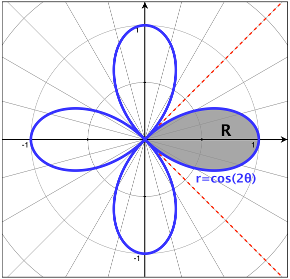

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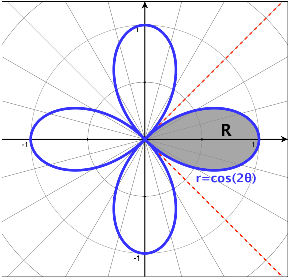

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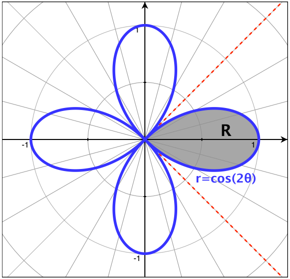

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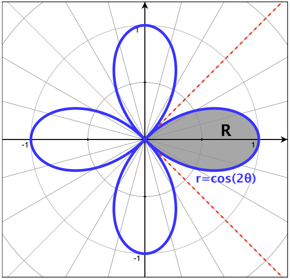

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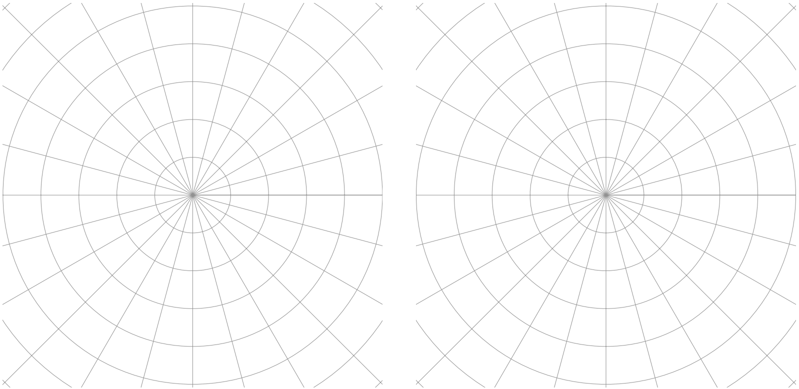
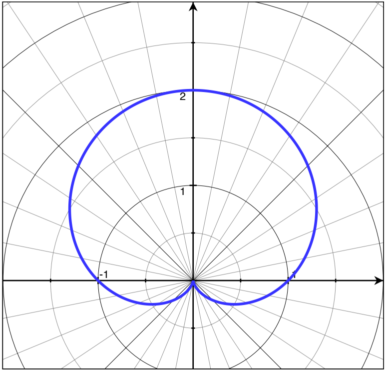
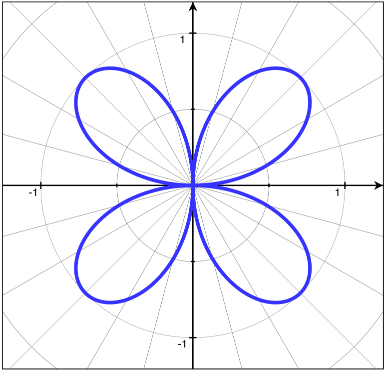
You try

For $r = r(\theta)$ positive, $\boxed{A = \int_a^b \frac{1}{2} r^2 \, d\theta}$

1. Calculate the area enclosed by the cardioid $r = 1 + \sin(x)$.
2. Graph the curve $r = \sin(2\theta)$ and calculate the area enclosed by this curve. (Careful! The function $\sin(2\theta)$ isn't always positive! Look at the picture and decide how to break up the problem.)



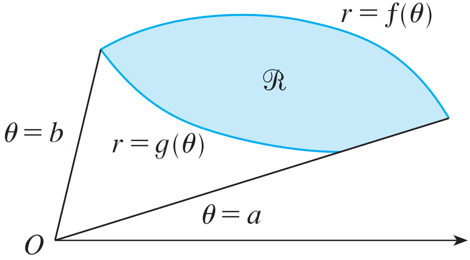




$$\int_0^{2\pi} \tfrac{1}{2}(1+\sin(\theta))^2\ d\theta = \tfrac{3}{2}\pi \qquad 4\int_0^{\pi/2} \tfrac{1}{2}(\sin(2\theta))^2\ d\theta = \pi/2$$

Area beween curves

Let R be the region bounded between curves $r = f(\theta)$ and $r = g(\theta)$ (where $0 \leq g(\theta) \leq f(\theta)$) and rays $\theta = a$ and $\theta = b$.



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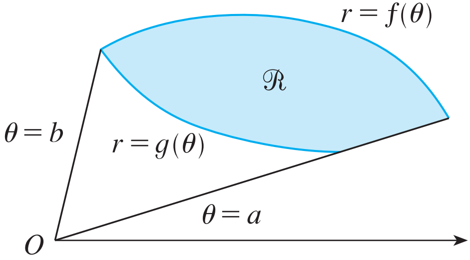

The area of R is given by

$$A = \int_a^b \frac{1}{2} r_{\text{out}}^2 \; d\theta - \int_a^b \frac{1}{2} r_{\text{in}}^2 \; d\theta$$

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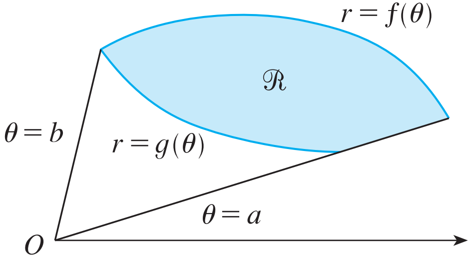

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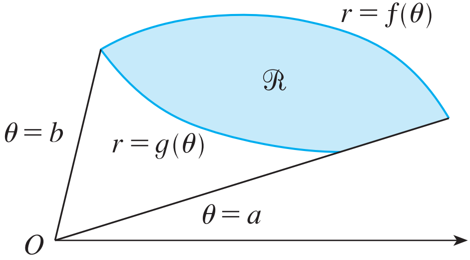

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Area beween curves

For $0 \leq r_{\text{in}} \leq r_{\text{out}}$, $$\boxed{A = \int_a^b \frac{1}{2}((r_{\text{out}}^2 - r_{\text{in}}^2))\ d\theta}$$

Example: Find the area of the region that lies inside the circle $r = 3\sin(\theta)$ and outside the cardioid $r = 1 + \sin(\theta)$.

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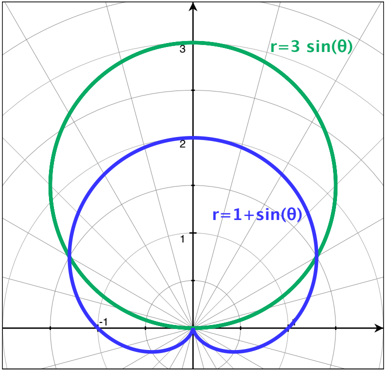

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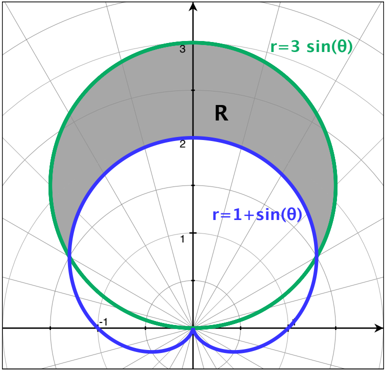

Next, identify the region and the bounds.

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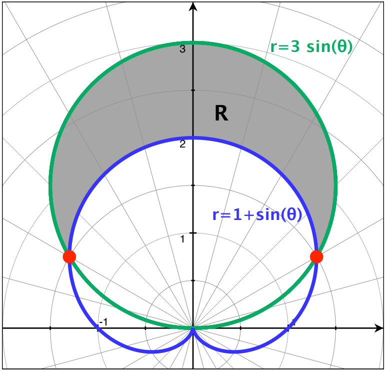

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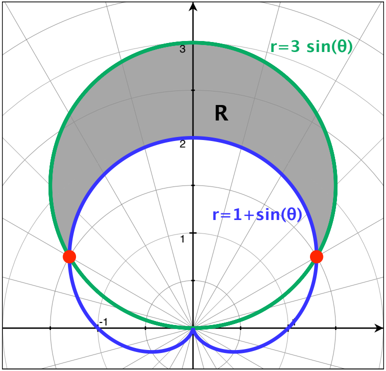

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Intersection points:

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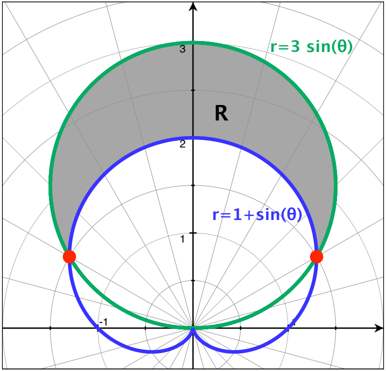

Next, identify the region and the bounds.

Intersection points:

$3\sin(\theta) = 1 + \sin(\theta)$, so $\sin(\theta) = \frac{1}{2}$

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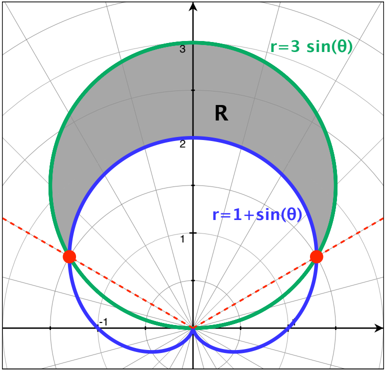

Next, identify the region and the bounds.

Intersection points:

$3\sin(\theta) = 1+\sin(\theta)$, so $\sin(\theta) = \frac{1}{2}$,

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Area beween curves

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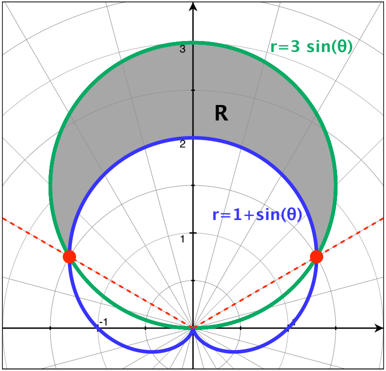

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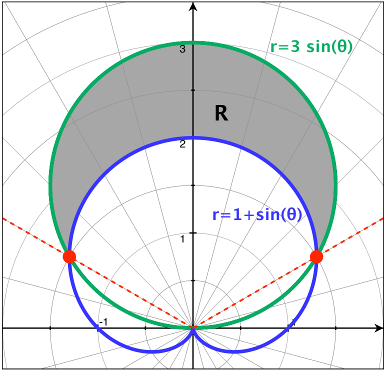

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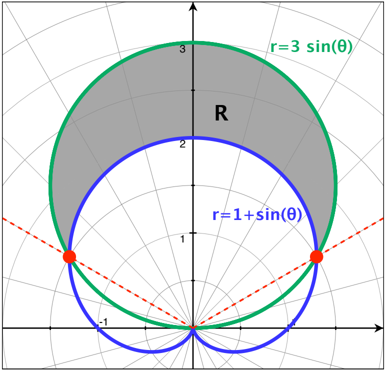

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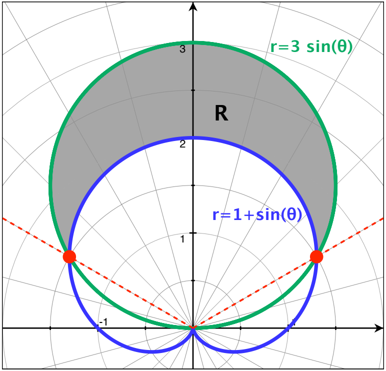

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(Check! Are both r's positive over the interval? If not, do we have to fix anything?)

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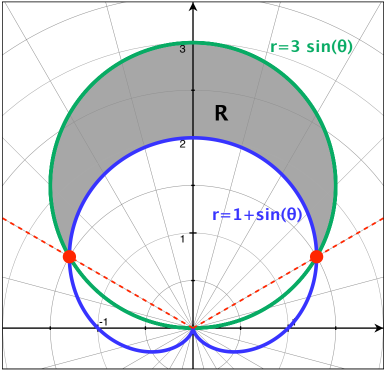

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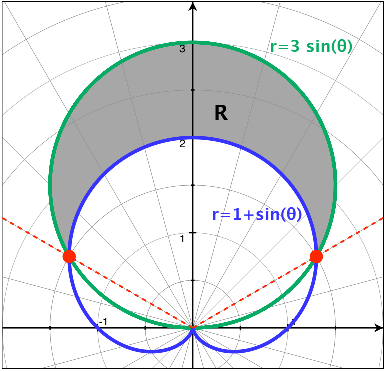

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$r_{\text{out}} = 3\sin(\theta)$, $\quad r_{\text{in}} = 1 + \sin(\theta)$

(Check! Are both r's positive over the interval? If not, do we have to fix anything?) ✓

So $A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}((3\sin(\theta))^2 - (1 + \sin(\theta))^2)\ d\theta$

You try:

For $r = r(\theta)$ positive, $\boxed{A = \int_a^b \frac{1}{2} r^2 \, d\theta}$

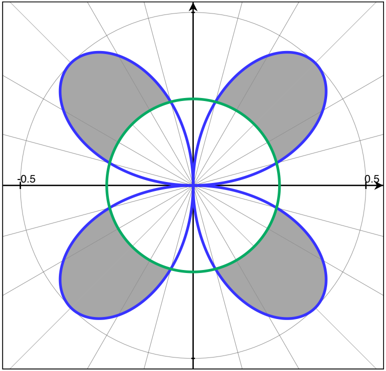
For $0 \leq r_{\text{in}} \leq r_{\text{out}}$, $\boxed{A = \int_a^b \frac{1}{2}((r_{\text{out}}^2 - r_{\text{in}}^2)) \, d\theta}$

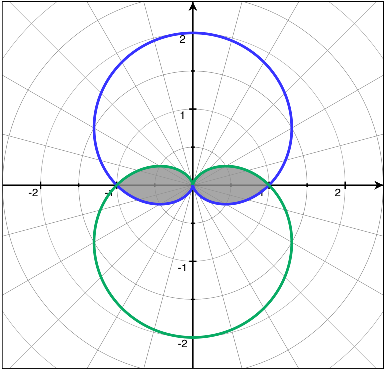
1. We showed that the area of the region that lies inside the circle $r = 3\sin(\theta)$ and outside the cardioid $r = 1 + \sin(\theta)$ is given by the integral

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}((3\sin(\theta))^2 - (1 + \sin(\theta))^2) \, d\theta.$$

   Check that this integral evaluates to π.

2. For the following regions, (i) graph the functions and identify the region, (ii) decide how to break your integral up using the above area formulas, (ii) calculate the area of the region.
   (a) Let R be the region inside $r = (1/2)\sin(2\theta)$ and outside $r = 1/4$.
   (b) Let R be the region enclosed by both $r = 1 + \sin(\theta)$ and $r = 1 - \sin(\theta)$.


-0.5
0.5


2
1
-1
-2
-2
2

Arc length (again!!)

Again! We start with

$$\ell = \int d\ell, \qquad \text{where } d\ell = \sqrt{dx^2 + dy^2}.$$

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When we had $x(t)$ and $y(t)$, we multiplied by dt/dt to get

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so multiply by $d\theta/d\theta$ to get

$$\boxed{d\ell = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta}$$

(exactly the same as the t case above)

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$$\frac{dx}{d\theta} = \frac{d}{d\theta} r(\theta)\cos(\theta) = \boxed{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}, \text{ and}$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} r(\theta)\sin(\theta) = \boxed{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}.$$

Arc length of parametric curves

$$\boxed{\ell = \int_{\theta=a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta}, \quad \text{where} \quad \begin{array}{l} x(\theta) = r(\theta)\cos(\theta), \\ y(\theta) = r(\theta)\sin(\theta). \end{array}$$

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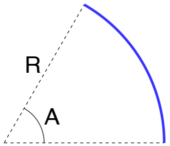

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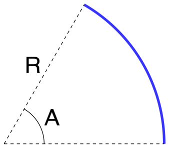

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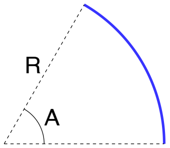

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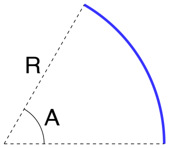

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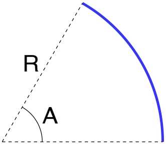

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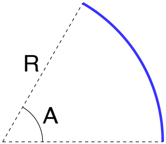

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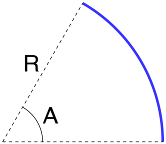

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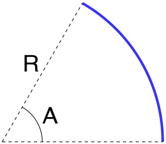

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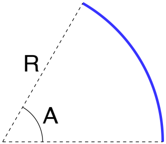

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You try

$$\boxed{\ell = \int_{\theta=a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}, \text{ where } \begin{array}{l} x(\theta) = r(\theta)\cos(\theta), \\ y(\theta) = r(\theta)\sin(\theta). \end{array}$$

1. Set up and simplify an integral that gives the length of the curve $r = \theta$ from $\theta = \pi/2$ to $\theta = \pi$.
2. Calculate the length of the curve $r = \cos(\theta)$ from $\theta = 0$ to $\theta = \pi$. (Hint: after doing derivatives, before setting up, recall double angle formulas.)
3. Calculate the length of the curve $r = e^{\theta}$, from $\theta = 1$ to $\theta = 2$.