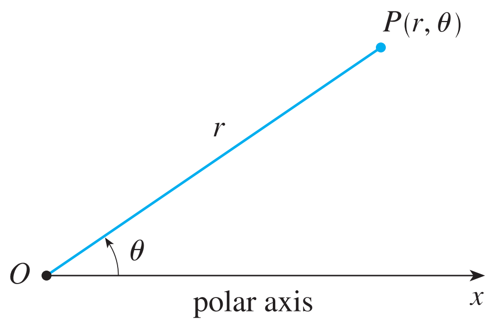
Polar coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers.
Cartesian coordinate system: start with x and y axes (perpendicular); points given by (x, y), where x is the the distance to the right of the y-axis, and y is the distance up from the x-axis.

Polar coordinate system: start with positive x-axis from before; points given by (r, θ), where r is the distance from the origin, and θ is the angle between the positive x- axis and a ray from the origin to the point, measuring counter-clockwise as usual.



Polar coordinates

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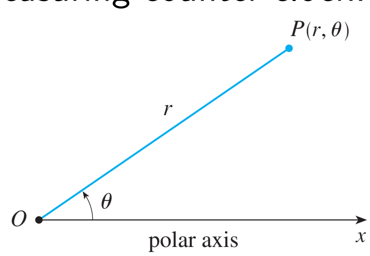
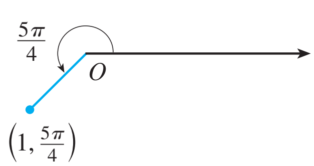
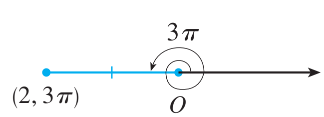


For example,

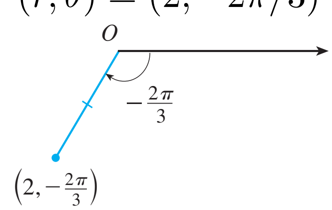
$(r, \theta) = (1, 5\pi/4)$



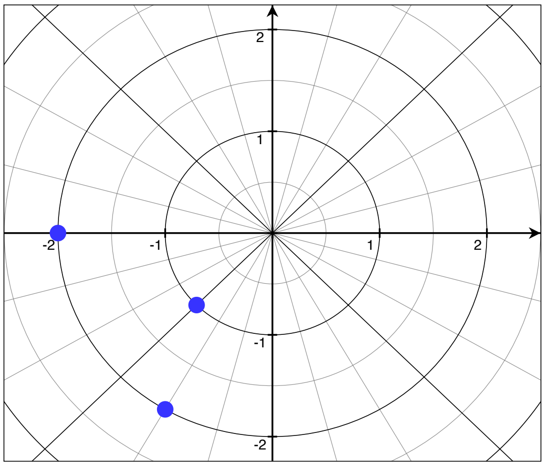
$(r, \theta) = (2, 3\pi)$



$(r, \theta) = (2, -2\pi/3)$



Polar axes



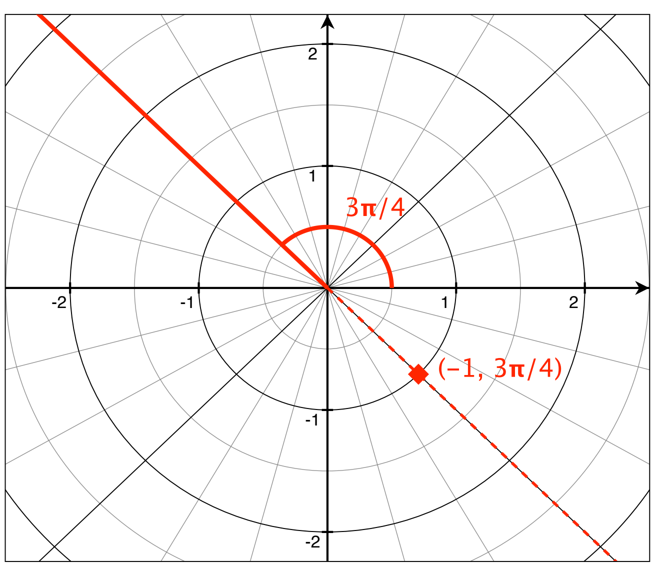
We plotted:
$(1, 5\pi/4), (2, 3\pi), (2, -2\pi/3)$

You try:
$(1/2, \pi/6), (3/2, 2\pi/3)$
$(2, -\pi/4), (1, -5\pi/4)$

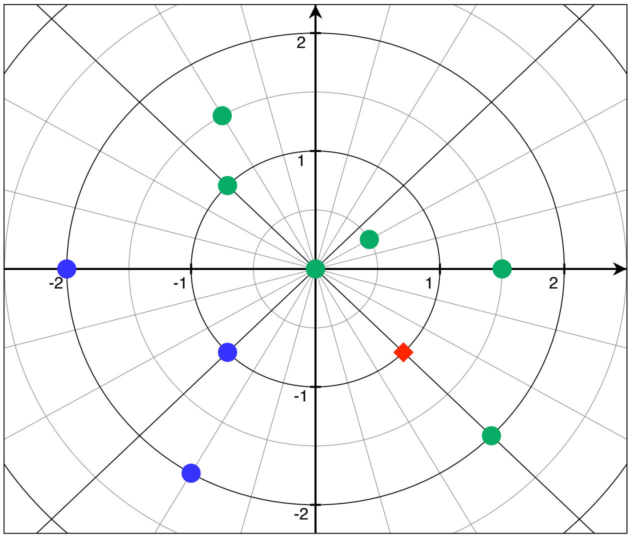
Polar axes

Negative radius: If r is negative, move backwards along the ray of angle θ.

For example, plot $(r, \theta) = (-3, 3\pi/4)$.



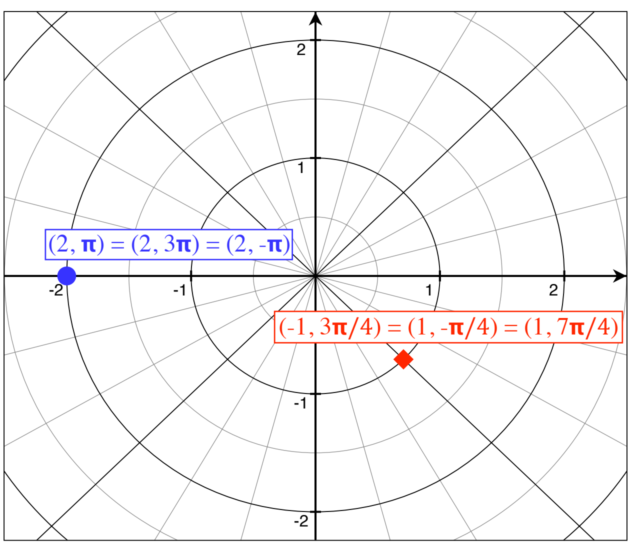
Polar axes



We plotted:
$(1, 5\pi/4), (2, 3\pi), (2, -2\pi/3)$
$(1/2, \pi/6), (3/2, 2\pi/3)$
$(2, -\pi/4), (1, -5\pi/4)$
$(-1, 3\pi/4)$

You try:
$(-1, 0), (-2, \pi/3),$
$(-3/2, -\pi/4)$

Polar axes



Notice we have some relations:

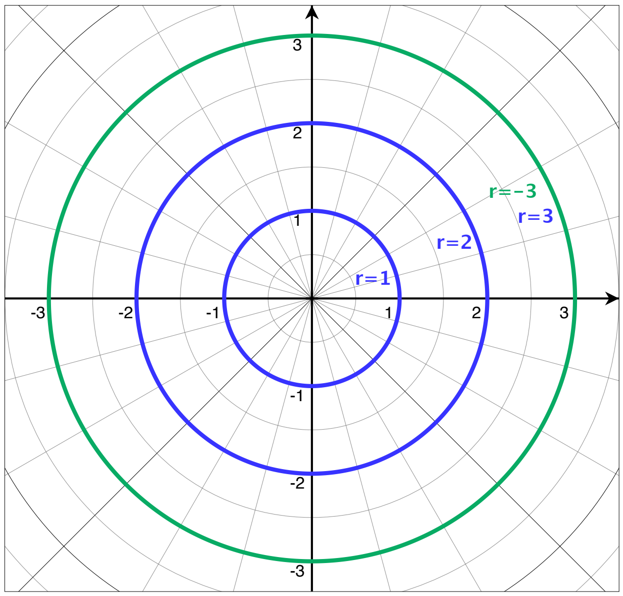
$$(r, \theta) = (r, \theta + k \cdot 2\pi) \quad \text{for } k = 0, \pm 1, \pm 2, \dots;$$
$$(-r, \theta) = (r, \theta \pm \pi).$$

Graphing

Just like we used to graph $y = f(x)$ or $x = g(y)$, now we graph things like $r = f(\theta)$ or $\theta = g(r)$.

Constant graphs:

$r =$ constant



Note:
$r = c$
is the same as
$r = -c$

Graphing

Just like we used to graph $y = f(x)$ or $x = g(y)$, now we graph things like $r = f(\theta)$ or $\theta = g(r)$.

Constant graphs:

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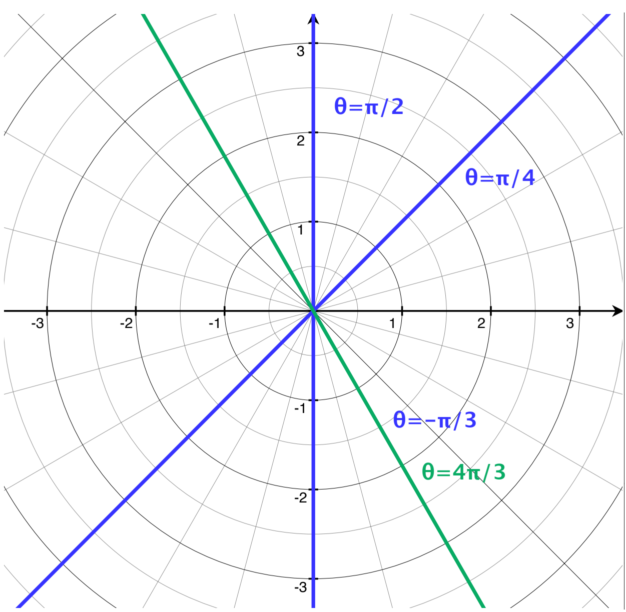
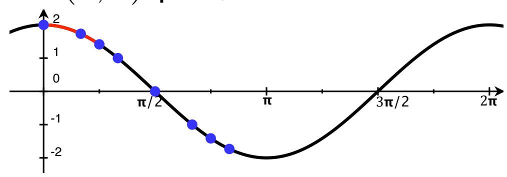


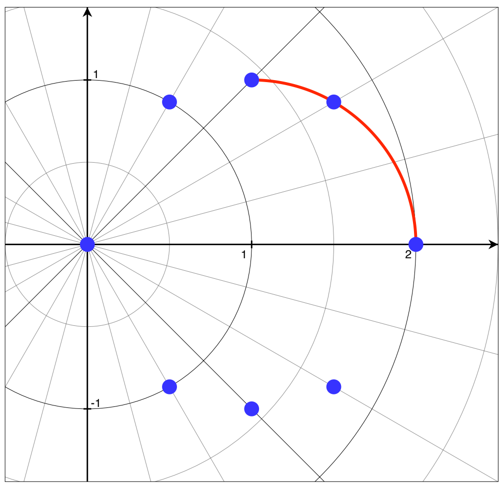
Note:
$\theta = c$
is the same as
$\theta = c + 2k\pi$

Graph the function $r = 2\cos(\theta)$.

First, on a Cartesian (θ, r) plot, this function looks like



(1) Plot points, (2) piece together segments



θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	2

Graph the function $r = 2\cos(\theta)$.

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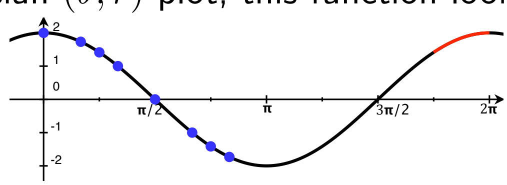
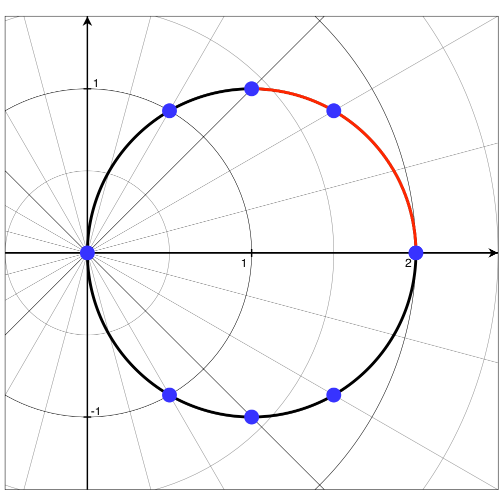


(1) Plot points, (2) piece together segments,
(3) stop at the end of one full period.



θ	r
0	2
$\pi/6$	$\sqrt{3}$
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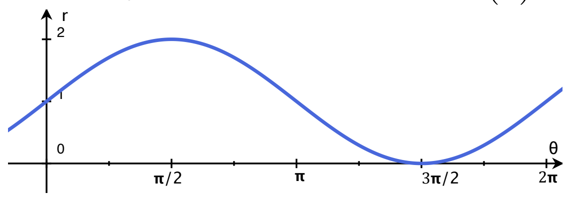
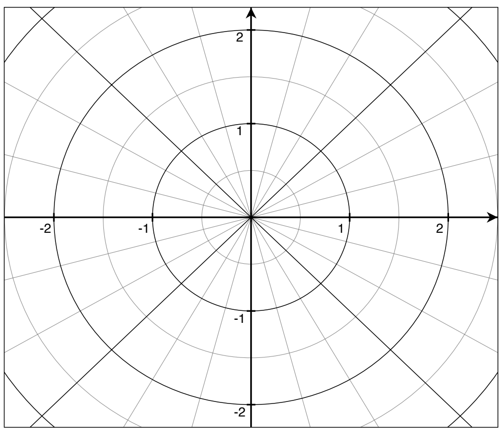
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	2

You try:

Note that on a θ-r axis, the curve $r = 1 + \sin(\theta)$ looks like



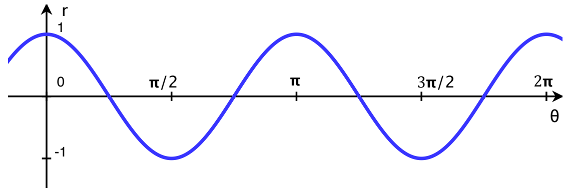
Sketch a graph of $r = 1 + \sin(\theta)$ on an x-y axis by plotting points, and piecing together segments as in the last example.



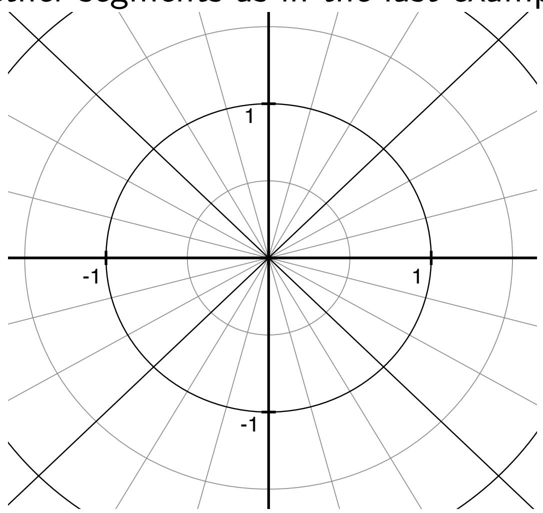
Note:
This graph is called the cardioid.

You try:

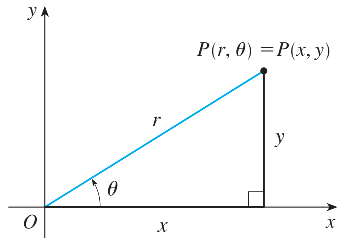
Note that on a θ-r axis, the curve $r = \cos(2\theta)$ looks like



Sketch a graph of $r = \cos(2\theta)$ on an x-y axis by plotting points, and piecing together segments as in the last example.



Converting between Cartesian and polar



$$\boxed{x = r\cos(\theta) \quad y = r\sin(\theta)}$$

To get back, we must solve for r and θ. To solve for r, we have the Pythagorean identity:

$$x^2 + y^2 = (r\cos(\theta))^2 + (r\sin(\theta)) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2 \cdot 1.$$

So

$$\boxed{r = \sqrt{x^2 + y^2}}.$$

To solve for θ, we eliminate r by dividing:

$$y/x = (r\sin(\theta))/(r\cos(\theta)) = \sin(\theta)/\cos(\theta) = \tan(\theta).$$

So

$$\boxed{\theta = \arctan(y/x)}.$$

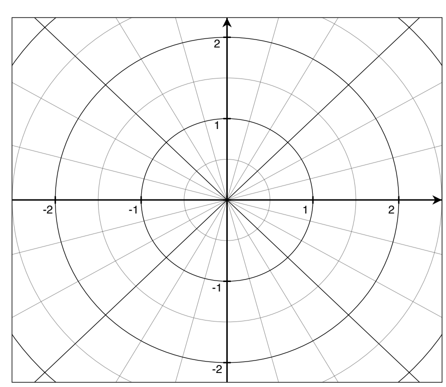
Converting between Cartesian and polar

$$\boxed{x = r\cos(\theta) \quad y = r\sin(\theta)}$$

$$\boxed{r = \sqrt{x^2 + y^2} \quad \theta = \arctan(y/x)}$$

You try: Convert the following points by filling out the rest of the table. Check by plotting.

(x, y)	(r, θ)
	$(2, \pi/3)$
	$(-1, \pi/4)$
$(1, 0)$	
$(1, -1)$	



Writing polar functions in terms of x and y

$$\boxed{x = r\cos(\theta) \quad y = r\sin(\theta)}$$

$$\boxed{r = \sqrt{x^2 + y^2} \quad \theta = \arctan(y/x)}$$

Example: $r = 2\cos(\theta)$. From $x = r\cos(\theta)$, we get $\cos(\theta) = x/r$. So

$$r = 2\cos(\theta) = 2x/r.$$

So

$2x = (2x/r)r = (r)r = r^2 = x^2 + y^2$. Thus $0 = x^2 - x + y^2$.

Completing the square gives

$$(x-1)^2 + y^2 = 1,$$

which is a unit circle shifted right by 1, as we saw.

You try: Write the following polar functions in terms of x and y.

(1) $r = 3$, (2) $\theta = \pi/3$, (3) $r = \sin(\theta)$.

Write the following Cartesian functions in terms of r and θ.

(1) $x^2 + y^2 = 4$, (2) $(x/3)^2 + y^2 = 1$, (3) $x = 2$.

Calculus with polar curves

Recall from last time, if I have a parametric curve $x = x(t)$, $y = y(t)$, then the slope of the line tangent to the curve plotted on an x-y axis is

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}.$$

We can use this result if we think about a polar curve as a parametric curve in parameter θ:

$$r = r(\theta) \quad \longleftrightarrow \quad \begin{aligned} x &= r(\theta)\cos(\theta) \\ y &= r(\theta)\sin(\theta) \end{aligned}$$

Example: $r = e^{\theta}$ is the same as the parametric curve

$$x = e^{\theta}\cos(\theta), \quad y = e^{\theta}\sin(\theta).$$

So now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\frac{d}{d\theta}(r(\theta)\sin(\theta))}{\frac{d}{d\theta}(r(\theta)\cos(\theta))} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}.$$

Calculus with polar curves

$$\boxed{\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\frac{d}{d\theta}(r(\theta)\sin(\theta))}{\frac{d}{d\theta}(r(\theta)\cos(\theta))} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}.}$$

Example: Let $r = 1 + \sin(\theta)$, the cardioid. We have $\frac{dr}{d\theta} = \cos(\theta)$, so that

$$\frac{dy}{d\theta} = \cos(\theta)\sin(\theta) + (1 + \sin(\theta))\cos(\theta) = \boxed{\cos(\theta)(1 + 2\sin(\theta))}$$

$$\frac{dx}{d\theta} = \cos(\theta)\cos(\theta) + (1+\sin(\theta))\sin(\theta)$$
$$= (1 - \sin^2(\theta)) + (1 + \sin(\theta))\sin(\theta) = \boxed{(1 + \sin(\theta))(1 - 2\sin(\theta))}.$$

So $$\boxed{\frac{dy}{dx} = \Big(\cos(\theta)(1 + 2\sin(\theta))\Big) / \Big(1 + \sin(\theta))(1 - 2\sin(\theta)\Big)}$$

You try: For what θ are the tangent lines to this cardioid horizontal? vertical?

$r = 1 + \sin(\theta)$