Warm up

For each of the following parametric curves,

(a) make a table of x and y values for 4 values of t, and plot those values on a x-y-axis;

(b) evaluate the range of x and y;

(c) write an expression relating x and y, eliminating the variable t;

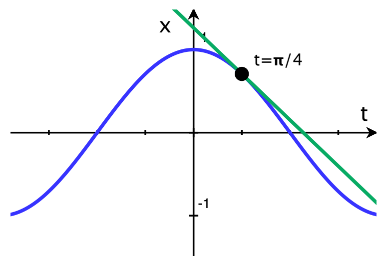
(d) sketch the parametric on the x-y-axis, keeping in mind the ranges you calculated in part (b).

1. $x = t + 1$, $y = t^2$;
2. $x = \cos(t)$, $y = \cos^3(t)$;
3. $x = 2^t$, $y = 2^{2t} + 1$;
4. $x = \cos(t)$, $y = 2\sin(t)$.

Calculus with parametric curves

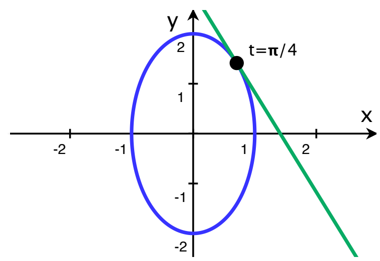
Say you've got a parametric curve $x(t)$, $y(t)$. If you graph $x(t)$ versus t, then the slope of the tangent line at any point on that curve is given by $x'(t)$. For example,

$$x(t) = \cos(t)$$
$$x'(t) = -\sin(t)$$
$$x'(\pi/4) = -\sqrt{2}/2$$



But what about if I graph y versus x? For example,

$$x(t) = \cos(t)$$
$$y(t) = 2\sin(t)$$
slope at $t = \pi/4$?



Calculus with parametric curves

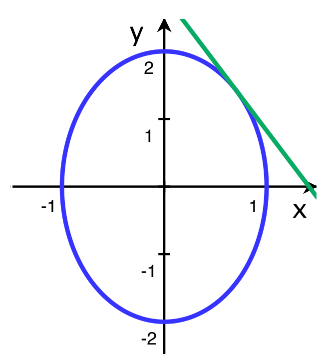
Given a parametric curve $x(t)$, $y(t)$, chain rule says $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. So

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)}.$$

Example: $x(t) = \cos(t)$, $y(t) = 2\sin(t)$.

$$\frac{dx}{dt} = -\sin(t) \quad \frac{dy}{dt} = 2\cos(t)$$

t	$\frac{dx}{dt}$	$\frac{dy}{dt}$	$\frac{dy}{dx}$
$\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}$	-2
$\pi/2$			
0			
$-\pi/4$			
$2\pi/3$			



Tangent lines

Recall point-slope form:

$$(y - y_0) = m(x - x_0), \quad \text{where} \quad m = dy/dx\big|_{(x_0,y_0)}.$$

For example: $x(t) = \cos(t)$, $y(t) = 2\sin(t)$, so that

$$\frac{dx}{dt} = -\sin(t) \quad \frac{dy}{dt} = 2\cos(t).$$

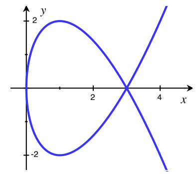
t	x	y	$\frac{dy}{dx}$	tangent line:
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}$	-2	$y - \sqrt{2} = -2(x - \sqrt{2}/2)$
$\pi/2$	0	2	0	
0	1	0	undef.	
$-\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}$	2	
$2\pi/3$	$-\sqrt{3}/2$	1	$2/\sqrt{3}$	

You try

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)}.$$

Consider the curve

$$x(t) = t^2,$$
$$y(t) = t^3 - 3t.$$



1. Calculate dx/dt, dy/dt, and dy/dx.
2. What are the three values of t where $y = 0$? Notice that two of the values have the same x-coordinate. What are the slopes of the two tangent lines through that point? What are the tangent lines' equations?
3. At what points is the tangent line horizontal? What is the tangent line's equation?
4. At what points is the tangent line vertical? What is the tangent line's equation?
5. For what values of t is dx/dt positive? is dy/dt positive? is dy/dx positive? Compare to the picture above.

Second derivatives

The second derivative we want is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We saw that chain rule says

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \quad \text{i.e.} \quad \frac{d}{dt}y = \left(\frac{d}{dx}y\right)\frac{dx}{dt}.$$

Replacing y by $\frac{dy}{dx}$ gives

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\frac{dx}{dt} = \left(\frac{d^2y}{dx^2}\right)\frac{dx}{dt}.$$

So

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\bigg/\frac{dx}{dt}}.$$

Second derivatives

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\bigg/\frac{dx}{dt}}$$

Back to the example $x(t) = t^2$, $y(t) = t^3 - 3t$. We saw

$$dx/dt = 2t, \quad dy/dt = 3t^2 - 3 = 3(t^2 - 1),$$

$$\text{so} \quad dy/dx = (3t^2 - 3)/(2t) = \tfrac{3}{2}(t - t^{-1}).$$

So now

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\tfrac{3}{2}(t - t^{-1}) = \tfrac{3}{2}(1 + t^{-2});$$

so that

$$\frac{d^2y}{dx^2} = \tfrac{3}{2}(1 + t^{-2})/2t = \tfrac{3}{4}(t^{-1} + t^{-3}).$$

You try:

For each of the following parametric curves,

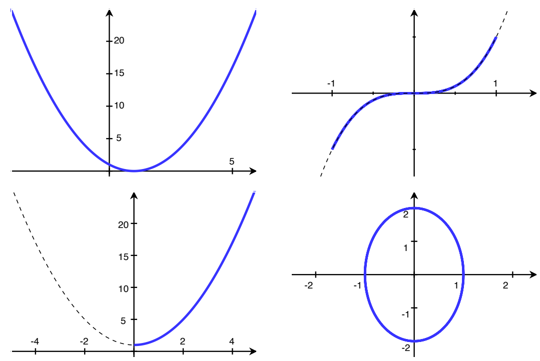
(a) calculate dx/dt, dy/dt, dy/dx, $\frac{d}{dt}(dy/dx)$, and d^2y/dx^2;

(b) for which values of t is dy/dx positive, negative, and 0?

(c) for which values of t is d^2y/dx^2 positive, negative, and 0?

(d) compare your results in (b) and (c) to the graphs we drew in the warmup.

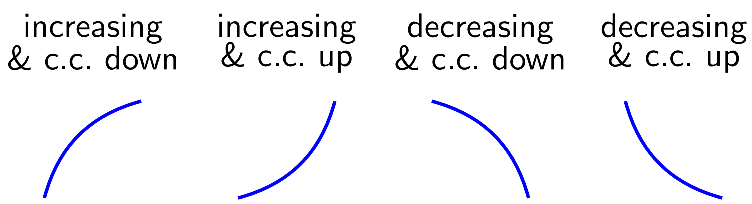
1. $x = t + 1$, $y = t^2$;
2. $x = \cos(t)$, $y = \cos^3(t)$;
3. $x = 2^t$, $y = 2^{2t} + 1$;
4. $x = \cos(t)$, $y = 2\sin(t)$.



Graphing curves

Recall, to sketch a curve $y = f(x)$ using calculus, we do the following:

1. Find when $f(x) = 0$, when $f(x)$ is undefined, and then when $f(x)$ is positive and negative.
2. Find asymptotes and limits as $x \to \infty$.
3. Find the critical points, i.e. when $f'(x) = 0$ or is undefined (where the tangent line is vertical or horizontal), and then when $f'(x)$ is positive and negative (when $f(x)$ is increasing or decreasing).
4. Find when $f''(x) = 0$ (possible inflection points), and then when $f''(x)$ is positive and negative (when $f(x)$ is concave up or down).
5. Piece the information together: plot important points and asymptotes, and piece shapes together corresponding to



You try

Sketch a graph of

$$x(t) = t^3 + 1 \quad y(t) = t^2 - t$$

on an x-y-axis by walking through the following steps:

1. For which t is $x(t) = 0$ or $y(t) = 0$? Plot those points.
2. As $t \to \infty$, what to $x(t)$ and $y(t)$ do? Same for $t \to -\infty$.
3. Calculate dx/dt, dy/dt, dy/dx. For what t is each positive, negative, zero, or undefined? Plot the points where changes happen.
4. Calculate $\frac{d}{dt}(dy/dx)$, and d^2y/dx^2. For what t is d^2y/dx^2 positive, negative, zero, or undefined? Plot the points where changes happen.
5. Put it all together!

Arc length of parametric curves

Recall, to calculate the length of a curve, we start with

$$\ell = \int d\ell, \qquad \text{where } d\ell = \sqrt{dx^2 + dy^2}.$$

When we had $y = f(x)$, we multiplied by dx/dx to get

$$d\ell = \sqrt{1 + (dy/dx)^2}\ dx.$$

When we had $x = f(y)$, we multiplied by dy/dy to get

$$d\ell = \sqrt{(dx/dy)^2 + 1}\ dy.$$

Now we start with $x(t)$ and $y(t)$, so we have dx/dt and dy/dt. Multiply by dt/dt to get

$$\begin{aligned} d\ell &= \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + dy^2}\ \frac{dt}{dt} \\ &= \sqrt{\frac{dx^2 + dy^2}{dt^2}}\ dt = \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}}\ dt \\ &= \boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}. \end{aligned}$$

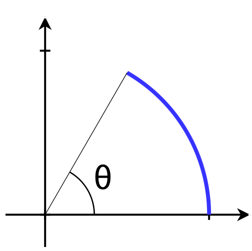
Arc length of parametric curves

$$\boxed{\ell = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$$

Example: Calculate the arc length of an arc of angle θ of a unit circle. The unit circle is given by the parametric curve

$$x(t) = \cos(t) \quad y(t) = \sin(t),$$

and an arc of angle θ is the curve traced from $t = 0$ to $t = \theta$:



$$dx/dt = -\sin(t)$$
$$dy/dt = \cos(t)$$
$$d\ell = \sqrt{(-\sin(t))^2 + (\cos(t))^2}\ dt = \sqrt{1}\ dt$$

So $\ell = \int_{t=0}^{\theta} d\ell = \int_{t=0}^{\theta} 1\ dt = t\Big|_{t=0}^{\theta} = \theta - 0 = \boxed{\theta}$.

You try

$$\boxed{\ell = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

1. Set up an integral that gives the length of the curve $x = t + 1$, $y = t^2$ from $t = 0$ to $t = 2$.
2. Set up an integral that gives the length of the curve $x = 2^t$, $y = 2^{2t} + 1$ from $t = 1$ to $t = 3$.
3. Set up an integral that gives the length of the curve $x = \cos(t)$, $y = \cos^3(t)$.
4. Set up an integral that gives the length of the curve $x = \cos(t)$, $y = 2\sin(t)$.