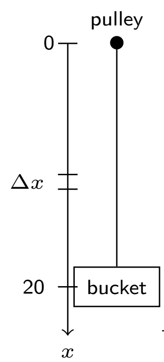
Work example: Leaky bucket

Suppose you lift a bucket of water straight up using a rope attached to a pulley. But as you lift the bucket, it leaks water at a constant rate.The bucket weights 2 lbs, the rope is 20 ft long and weights a total of 10 lbs. The rope is wound around the pulley at a rate of 2 ft/s. The bucket starts out holding 15 lb of water and leaks at a rate of 1/10 lb/s. How much work is required to lift the bucket to the top?

Answer: We do this problem in three parts, (1) the bucket, (2) the rope, and (3) the water.



(1) Bucket. The bucket exerts a force of 2 lbs, and is lifted 20 ft, so $\boxed{W_{\text{bucket}} = 2(20) \text{ ft-lbs}}$.

(2) Rope. Break the rope into vertical segments of length Δx. Each segment exerts a force of (10/20 lb/ft) Δx ft, and the segment of rope at height x gets lifted x ft.

$$W_{\text{rope}} = \int_0^{20} (10/20) x dx = \tfrac{1}{2}(x^2/2)\Big|_0^{20} = \tfrac{1}{4}(20)^2.$$

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$$\boxed{W_{\text{bucket}} = 2(20)} \qquad \boxed{W_{\text{rope}} = \tfrac{1}{4}(20)^2}$$

(3) Water. The work done to lift the water from height x to height $x - \Delta x$ is $f(x)\Delta x$, where $f(x) =$ weight of water remaining at position x, so that $W_{\text{water}} = \int_0^{20} f(x)dx$. As a function of time, starting from when the bucket begins to be lifted, the position of the bucket is $x = 20 - 2t$. So time, as a function of position, is $\boxed{t = 10 - \frac{1}{2}x}$. Also as a function of time, the weight of the bucket is $10 - (1/10)t$. So

$$f(x) = 10 - (1/10)t(x) = 10 - (1/10)(10 - \tfrac{1}{2}x) = \boxed{9 + \tfrac{1}{20}x}.$$

So

$$W_{\text{water}} = \int_0^{20} 9 + \tfrac{1}{20}x \, dx = (9x + \tfrac{1}{40}x^2)\Big|_0^{20} = \boxed{9(20) + \tfrac{1}{40}(20)^2}.$$

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$$\boxed{W_{\text{bucket}} = 2(20)} \qquad \boxed{W_{\text{rope}} = \tfrac{1}{4}(20)^2}$$

$$\boxed{W_{\text{water}} = 9(20) + \tfrac{1}{40}(20)^2}$$

So in total,

$$W = W_{\text{bucket}} + W_{\text{rope}} + W_{\text{water}} = \boxed{2(20) + \tfrac{1}{4}(20)^2 + 9(20) + \tfrac{1}{40}(20)^2}.$$

9.1 Parametric curves

In the water portion of the previous problem, position and weight started out as functions of time:

$$x(t) = 20 - 2t \qquad \text{and} \qquad f(t) = 10 - (1/10)t.$$

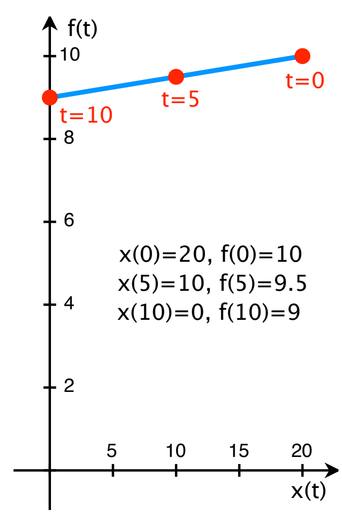
These are called parametric equations, with parameter t. Separately, they're just two functions of time. But together, they are coupled by their common parameter. We can thus graph f versus x by varying t.

To find the equation for f as a function of x, we solved x for t, and plugged that into f:

$$t = 10 - \tfrac{1}{2}x,$$

so

$$f = 10 - (1/10)(10 - \tfrac{1}{2}x) = 9 + \tfrac{1}{20}x.$$

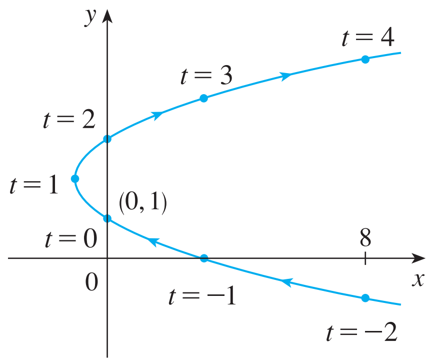


Example: Define the parametric curve by

$$x(t) = t^2 - 2t, \qquad y(t) = t + 1.$$

Plotting the curve: Pick a sample of values for t:

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



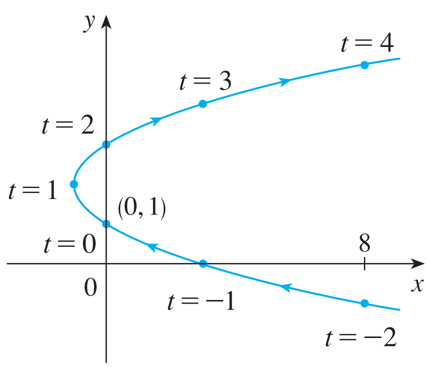
This curve is suited best writing x as a function of y, so solve for t in terms of y and plug in:

$$t = y - 1, \qquad \text{so } x = (y-1)^2 - 2(y-1) = (y-2)^2 - 1.$$

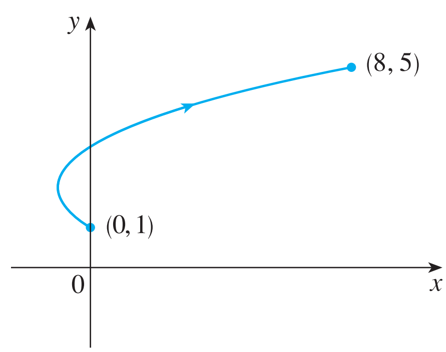
Example: Define the parametric curve by

$$x(t) = t^2 - 2t, \qquad y(t) = t + 1.$$

For all t:



For $0 \leq t \leq 4$:

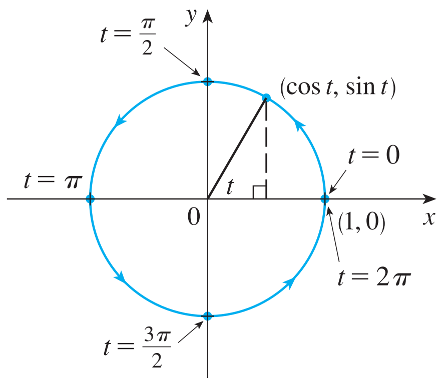


Writing the function just in terms of x and y loses some information. If we're thinking about the parametric function as a particle traveling on the x-y plane over time, we calculated that it traces the curve $x = (y-2)^2 - 1$, but it doesn't tell us what direction or how fast. Further, we have put no restriction on t.

Example: Unit circle.

$$x(t) = \cos(t) \quad y(t) = \sin(t), \quad 0 \le t \le 2\pi$$

Plotting points



This curve traces out a circle! (Recall the unit circle)

Converting to a function of just x and y:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.$$

You try: Graph and compare the following parametric curves to each other and the example above.

(1) $x(t) = \cos(2t), y(t) = \sin(2t), \quad 0 \le t \le 2\pi;$

(2) $x(t) = \cos(t/3), y(t) = \sin(t/3), \quad 0 \le t \le 2\pi.$

Graph transformations

Since a parametric curve gives the x and y coordinates separately, transformations are a little more straightforward.
Example: We saw that $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$ is the unit circle centered at the origin.

If I want a circle centered at the point $(2, 5)$, that's the same as shifting all the x-coordinates right by 3 and all the y-coordinates up by 5:

$$x(t) = \cos(t) + 2 \quad y(t) = \sin(t) + 5, \quad 0 \le t \le 2\pi.$$

If instead I still want a circle centered at $(0, 0)$, but I want its radius dilated to 3, I want to multiply the x and y coordinates all by 3:

$$x(t) = 3\cos(t) \quad y(t) = 3\sin(t), \quad 0 \le t \le 2\pi.$$

If I want a bigger circle that's also shifted, dilate first and then shift (just as before). A circle of radius r, centered at (a, b) is given by

$$x(t) = r\cos(t) + a \quad y(t) = r\sin(t) + b, \quad 0 \le t \le 2\pi.$$

Check:

$$(x-a)^2 + (y-b)^2 = (r\cos(t))^2 + (r\sin(t))^2 = r^2(\cos^2(t) + \sin^2(t)) = r^2. \checkmark$$

You try:

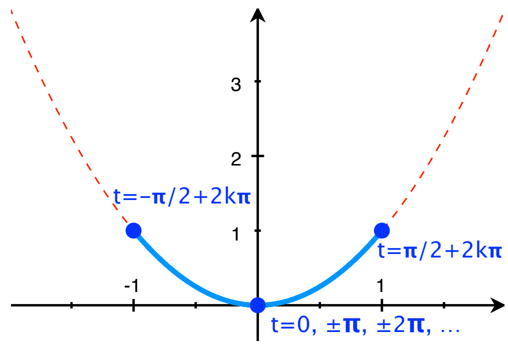
Sketch the following curves and give a formula for their shape just in terms of x and y.
(Hint: Think about graph transformations, and scaling or shifting either or both coordinates to get them to fit the pythagorean identity.) Give an example of a domain for t that would trace the curve exactly once.

1. $x(t) = 2\cos(t)$, $y(t) = \sin(t)$.
2. $x(t) = \cos(t)$, $y(t) = 3\sin(t)$.
3. $x(t) = \cos(t)$, $y(t) = \sin(-t)$.
4. $x(t) = 5\cos(2t)$, $y(t) = 3\sin(2t)$.
5. $x(t) = \cos(t)$, $y(t) = \sin(-t)$.
6. $x(t) = \sin(t)$, $y(t) = \cos(t)$.
7. $x(t) = 2\cos(t) + 1$, $y(t) = 3(\sin(t) - 4)$.
8. $x(t) = 5\cos(-t) + 1$, $y(t) = 2\sin(t) + 5$.

Example: Sketch

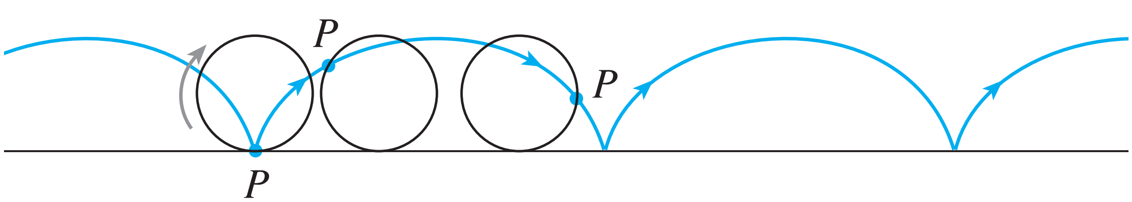
$$x(t) = \sin(t), \quad y(t) = \sin^2(t).$$

We could solve for t from one and plug it into the other. But as a shortcut, it's clear to see that $y = \sin^2(t) = x^2$. So this curve appears to be a parabola.

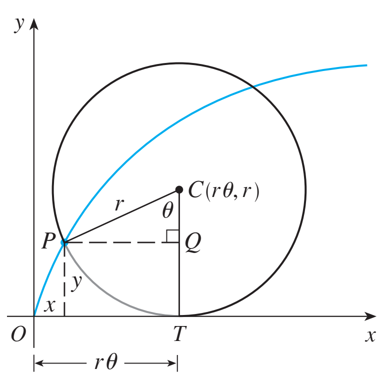


But $-1 \leq \sin(x) \leq 1$, so the x values can't go outside these bounds. This is actually a curve traced out by a particle bouncing back and forth between $(-1, 1)$ and $(1, 1)$ along the curve $y = x^2$.

Take a wheel or radius r, and mark one point on its boundary. Now roll that wheel, and trace the path that the marked point takes:



This curve is called a cycloid. To calculate its formula, we'll use a param. curve:



Parameter: θ, the rot'l angle of circle.

Center: The edge of the circle has all touched the ground. So the distance of the center from the y-axis is the arc length of the circle with angle θ. So the center is at $C = (r\theta, r)$.

Then

$$x = |OT| - |PQ| = r\theta - r\sin\theta,$$
$$y = |TC| - |QC| = r - r\cos\theta.$$