7.6 "Work" from physics

Last time: Work W means the the effort it takes to move an object. This is the amount of force exerted, times the distance d moved. If $f(x)$ is the force exerted as a function of position x, then the amount of work done moving the object from $x = a$ to $x = b$ is

$$W = \int_a^b f(x)\ dx.$$

To set up these problems (mostly word problems), your goal is to calculate the function $f(x)$ and then integrate.

If the word problem involves a spring, you want to use Hooke's law:

$f(x) = kx$, where k is a constant particular to the spring.

Otherwise, you usually want to use Newton's second law of motion:

$f(x) = ma$, where a is usually the accel. of gravity.

Note: in the metric system, $g = 9.8$ m/s^2. In the US system, "pounds" force of a mass under gravity on Earth.

Examples from last time:

Example 1: When a particle is moved by a force of $f(x) = x^2 + 2x$ pounds (x in feet), how much work is done by moving is from $x = 1$ to $x = 3$?

Answer: Here, we're just given the force equation, so we just integrate:

$$W = \int_{x=1}^3 f(x)dx = \int_1^3 x^2 + 2x\ dx = \tfrac{50}{3}\text{ft-lb}.$$

Example 2: Suppose a force of 40 N is requires to hold a spring 5cm from its equilibrium. How much work is done in stretching is from 5cm to 8 cm from equilibrium?

Answer: Here, we're *not* given the force equation, so we have to calculate it. It's a spring problem, so we want to use Hooke's law, $f(x) = kx$. Step 1 is calculate k. Step 2 is plug in and integrate.

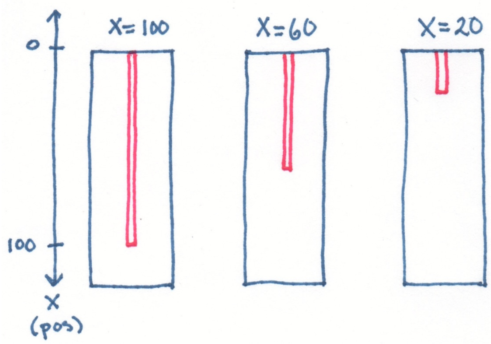
(1) $f(x) = 40$ N $= kx = k0.05$ m, so $k = 40/.05 = 800$.

(2) $W = \int_{0.05}^{0.08} 800x\ dx = 1.56$ J.

Example 3

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Answer: Use Newton's 2nd law: $f(x) = ma$. We want this to end up a function of *position*. Draw a picture!



We're using the US system, so $f(x) = m(x)$. Also, as I pull the cable up, the weight of what's left changes. So $f(x)$ is a percentage of the total weight of the cable. Percentage: $x/100$.

$$f(x) = \left(\frac{x}{100}\right) 200 \text{ lb} = 2x \text{ lb}.$$

So

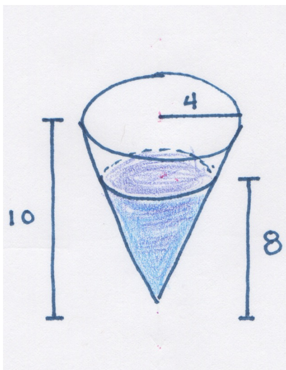
$$W = \int_0^{100} 2x \, dx = 100^2 \text{ft-lb}.$$

Example 4

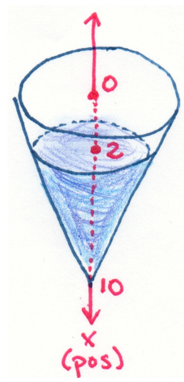
A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is $\rho = $1000 kg m^3.)

Answer: Use Newton's 2nd law: $f(x) = ma$. Again, we want this to end up a function of *position*. Draw a picture!

Starting picture:



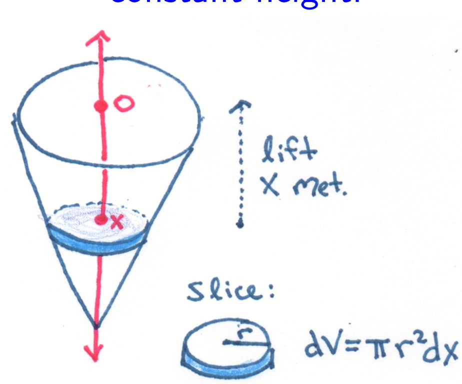
Add coordinate axes:



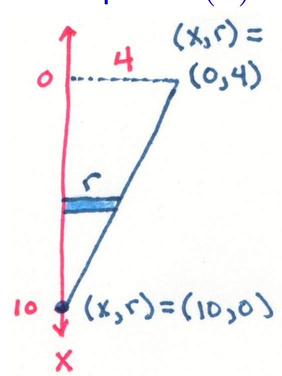
Think of the work as lifting each water molecule up to the top of the tank. Of course, all the water that's at a fixed height takes the same amount of work per molecule. Slice it horizontally!

Example 4

Slices with constant height:



Cut vertically to compute $r(x)$:



$$dF = g\ dM = g\rho\ dV$$
$$= (9.8)(1000)(\pi r^2(x)dx)$$

similar triangles: $r(x)/(10-x) = 4/10$

So $r(x) = \frac{2}{5}(10-x)$

Thus $dF = (9.8)(1000)(\pi)\left(\frac{2}{5}(10-x)\right)^2 dx$, so that

$$W = \int_2^{10} (9.8)(1000)(\pi)\left(\tfrac{2}{5}(10-x)\right)^2 dx.$$

You try

Set up the following problems.

1. A chain lying on the ground is 5 m long, and its mass is 100 kg. How much work is required to raise one end of the chain to a height of 7 m?
2. A rectangular swimming pool has sides of length 20 ft and 30 ft, and a constant height of 6 ft. It is filled with water to a depth of 5 ft. How much work is required to pump all of the water out over the side (top)? (Use the fact that water weighs 62.5 lb ft^3.)