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To set up these problems (mostly word problems), your goal is to calculate the function $f(x)$ and then integrate.

If the word problem involves a spring, you want to use Hooke's law:

$f(x) = kx,$ where k is a constant particular to the spring.

Otherwise, you usually want to use Newton's second law of motion:

$f(x) = ma,$ where a is usually the accel. of gravity.

Note: in the metric system, $g = 9.8$ m/s^2. In the US system, "pounds" force of a mass under gravity on Earth.

Examples from last time:

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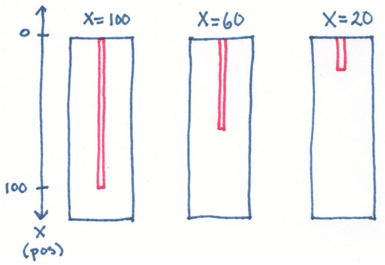

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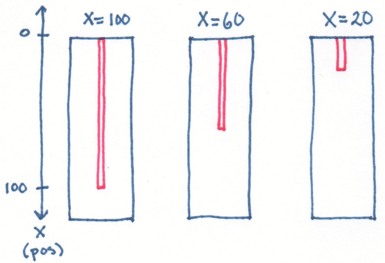

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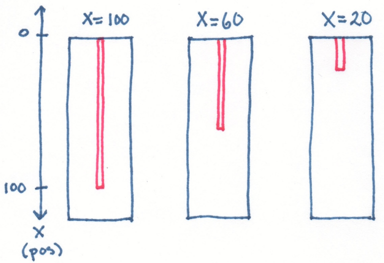

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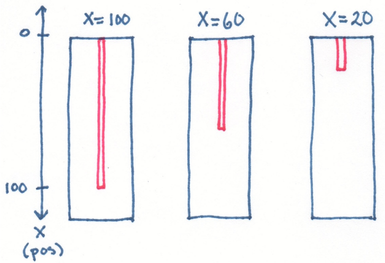

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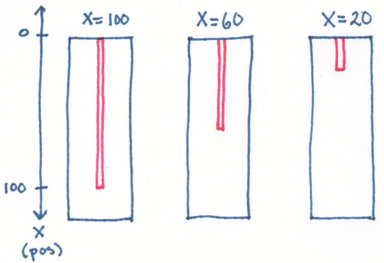

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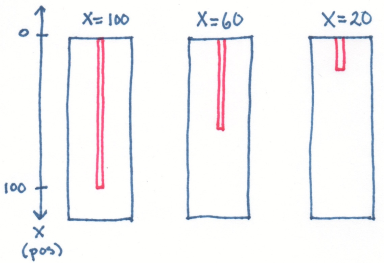

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A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is $\rho =$1000 kg m^3.)

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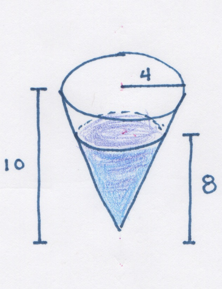

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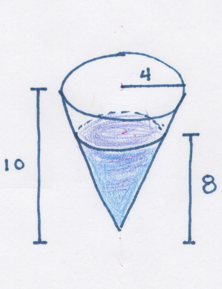
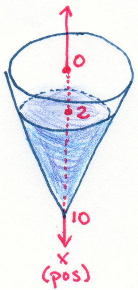


Add coordinate axes:



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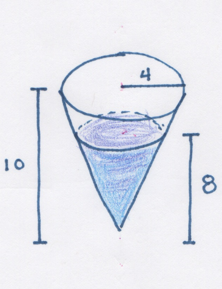

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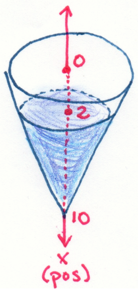

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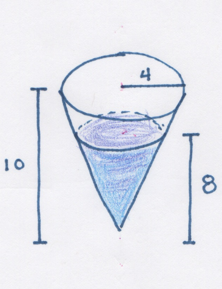

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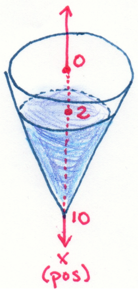

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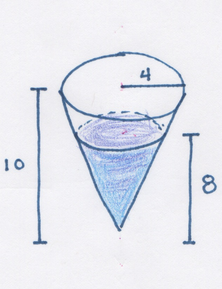

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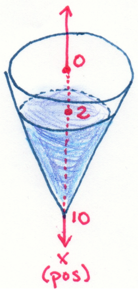

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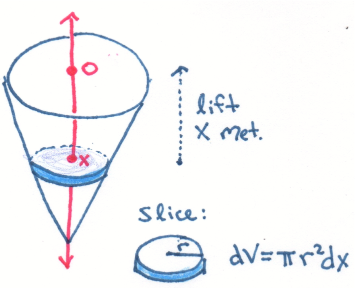
Slices with constant height:



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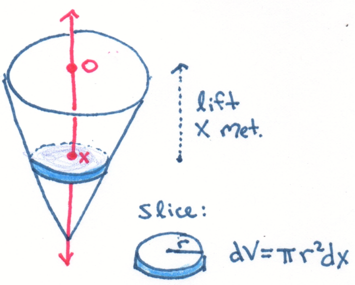

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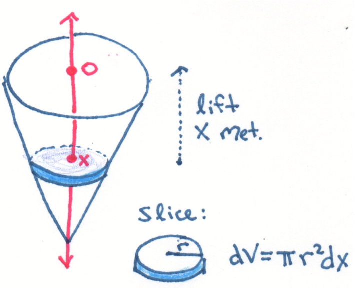

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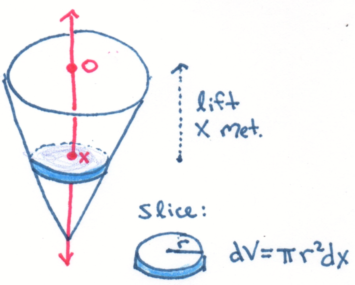

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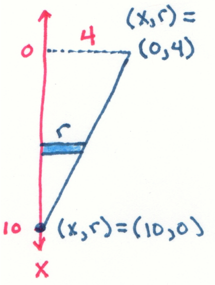

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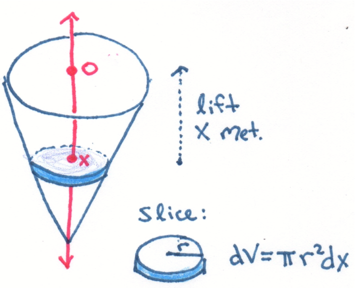

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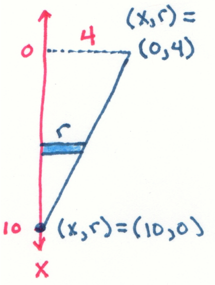

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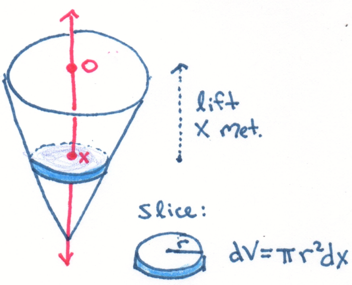

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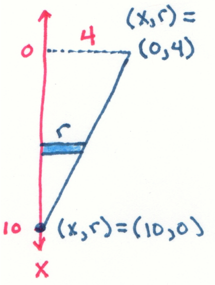

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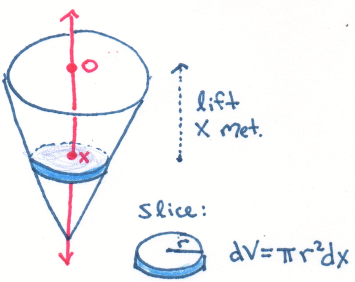

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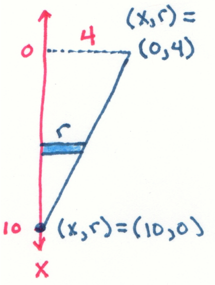

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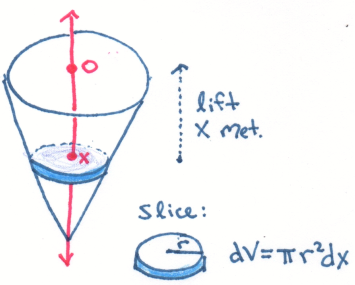

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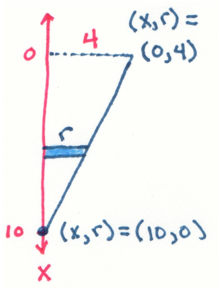

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Thus $dF = (9.8)(1000)(\pi)\left(\frac{2}{5}(10-x)\right)^2 dx$, so that

$$W = \int_2^{10} (9.8)(1000)(\pi)\left(\frac{2}{5}(10-x)\right)^2 dx.$$

You try

Set up the following problems.

1. A chain lying on the ground is 5 m long, and its mass is 100 kg. How much work is required to raise one end of the chain to a height of 7 m?
2. A rectangular swimming pool has sides of length 20 ft and 30 ft, and a constant height of 6 ft. It is filled with water to a depth of 5 ft. How much work is required to pump all of the water out over the side (top)? (Use the fact that water weighs 62.5 lb ft^3.)