# 7.5 Surface area

When we did areas, the basic slices were rectangles, with

$$\Delta A = h\Delta x \quad \text{or} \quad h\Delta y.$$

When we did volumes of revolution, the basic slices came from revolving rectangles around an axis. Depending on whether the rectangles were parallel or perpendicular to the axis, we got washers, with

$$\Delta V = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\Delta x \quad \text{or} \quad \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\Delta y,$$

or cylindrical shells, with

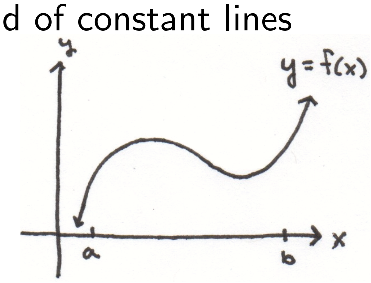
$$\Delta V = 2\pi r h\Delta x \quad \text{or} \quad 2\pi r h\Delta y.$$

Last time, we calculated arc length, the basic slices were line segments, with

$$\Delta \ell = \sqrt{1 + (\tfrac{dy}{dx})^2}\Delta x \quad \text{or} \quad \sqrt{1 + (\tfrac{dx}{dy})^2}\Delta y.$$

Today: instead of rotating filled in regions around an axis and calculating the volume of the shape, we will rotate curves around an axis and calculate the surface area of the shape.

The arc length approximation was a lot like when we approximated area with trapezoids, where each piece is a line with some slope, instead of constant lines

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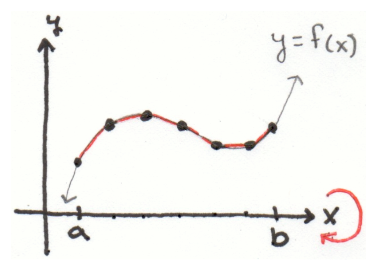
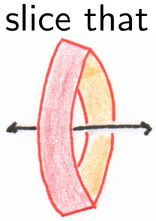
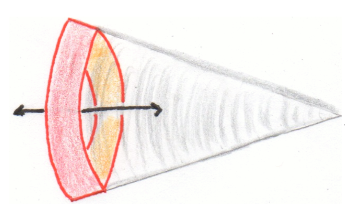



$$d\ell = \sqrt{1 + (\tfrac{dy}{dx})^2}dx$$

Now, if we rotate one of those segments around the $x$-axis, we get a slice that looks like



, which is part of the cone



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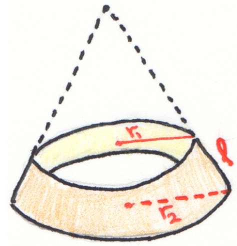
So we need the surface area of a cross-section of a cone,

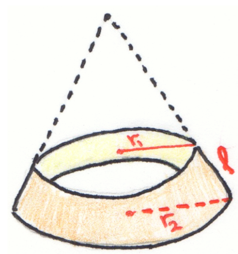



which depends on the lateral length $\ell$ of the slice, and the average of the radius $r_1$ of the small circle and the radius $r_2$ of the big circle. This surface area is given by (see section 7.5)

$$A = 2\pi\left(\tfrac{1}{2}(r_1 + r_2)\right)\ell$$

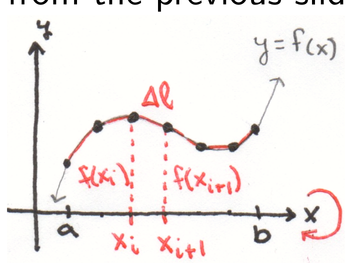
(i.e. the area of the circular cylinder whose height is the the lateral length $\ell$ and whose radius is the average of the two extreme radii).

So we need the surface area of a cross-section of a cone,




$$A = 2\pi\left(\tfrac{1}{2}(r_1 + r_2)\right)\ell$$
$$\Delta A = 2\pi\left(\tfrac{1}{2}(f(x_i) + f(x_{i+1})\right)\Delta\ell$$

In our slice from the previous slide,

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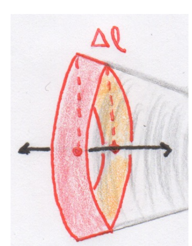



the lateral length is $\Delta\ell$ and the two radii are given by the height of the function at $x_i$ and $x_{i+1}$. But as $n \to \infty$,

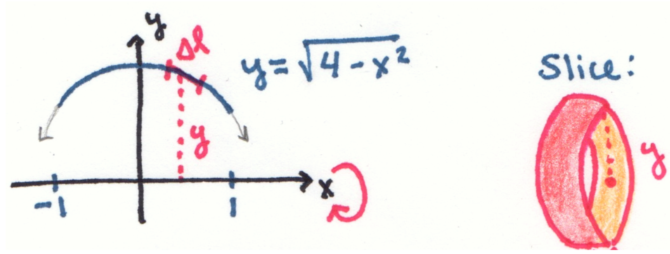
$\Delta\ell \to d\ell = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$ and $\frac{1}{2}(f(x_i) + f(x_{i+1})) \to f(x)$. So

$$\Delta A \to \boxed{dA = 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx}.$$

# Example

$$\boxed{dA = 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx}$$

Calculate the surface area of shape generated by rotating the curve $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$, around the $x$-axis:




Here, $f(x) = \sqrt{4 - x^2}$, so that $f(x) = \frac{-2x}{2\sqrt{4-x^2}} = \frac{x}{\sqrt{4-x^2}}$. Thus

$$1 + (f'(x))^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4 - x^2 + x^2}{4 - x^2} = \frac{4}{4 - x^2}.$$

So

$$A = \int_{x=-1}^{1} dA = \int_{-1}^{1} 2\pi\sqrt{4 - x^2}\sqrt{4/(4 - x^2)}\ dx$$

$$= \int_{x=-1}^{1} 2\pi(2)dx = 4\pi x\Big|_{-1}^{1} = 8\pi.$$

# You try:

Set up the following problems (simplify until you have something you can integrate, but don't finish the integration).

Calculate the surface area of shape resulting from...

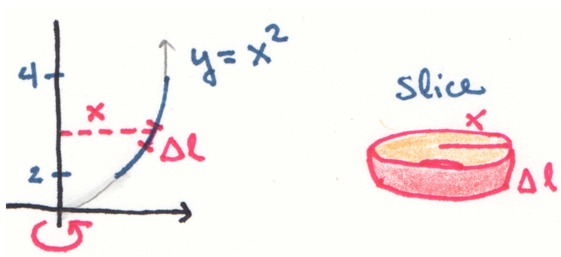
1. revolving $y = \sqrt{1 + e^x}$ for $0 \leq x \leq 1$ around the $x$-axis;
2. revolving $y = 1 + 2x^2$ for $1 \leq x \leq 2$ around the $x$-axis;
3. revolving $y = \sin(x)$ for $0 \leq x \leq \pi/4$ around the $x$-axis.

## Changing the axis of rev. doesn't change the variable:

We started with $dA = 2\pi r d\ell$.

If I revolve around the $y$-axis instead, I can still write $d\ell$ in terms of $x$. The difference is that the radius is the distance from the $y$ axis instead of the difference from the $x$-axis. So $r = x$!

Example: Revolve the curve $y = x^2$ for $1 \leq x \leq 2$ around the $y$-axis.




If I want my variable to be $x$, I have $r = x$, and $d\ell = \sqrt{1 + (dy/dx)^2}\ dx$, where

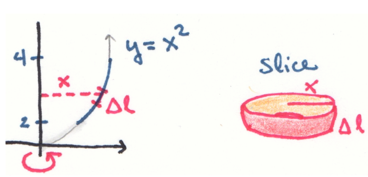
$$\frac{dy}{dx} = 2x, \quad \text{so} \quad 1 + (dy/dx)^2 = 1 + 4x^2.$$

## We can force a variable change

We started with $dA = 2\pi r d\ell$. I could choose to write $d\ell$ in terms of $y$ instead, as

$$d\ell = \sqrt{1 + (dx/dy)^2}\ dy.$$

Example again: Revolve the curve $y = x^2$ for $1 \leq x \leq 2$ around the $y$-axis.




Now if I want my variable to be $y$, I have $r = x = \sqrt{y}$, and $d\ell = \sqrt{1 + (dx/dy)^2}\ dx$, where

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}, \quad \text{so} \quad 1 + (dy/dx)^2 = 1 + \frac{1}{4y} = \frac{4y+1}{4y}.$$

Thus

$$A = \int_{y=1}^{4} dA = \int_1^4 2\pi\sqrt{y}\sqrt{\tfrac{4y+1}{4y}}\ dy = \int_1^4 \pi\sqrt{4y+1}\ dy$$
$$= \pi\tfrac{1}{4}\tfrac{2}{3}(4y+1)^{3/2}\Big|_1^4 = \tfrac{\pi}{6}((17)^{3/2} - 5^{3/2}) \quad \checkmark$$

# You try:

Consider the shape generated by revolving

$$y = \frac{1}{2}\sqrt{x-1} \quad 3 \leq x \leq 9$$

around the $x$-axis. Compute (if possible) the resulting surface area in two ways, using $x$ as the variable, and then using $y$ as the variable. Then do the same thing for rotating the same curve around the $y$-axis.

## 7.6 "Work" from physics

In physics, work $W$ means the the effort it takes to move an object. This is the amount of force exerted, times the distance $d$ moved. Newton's second law of motion describes force $F$ as mass $m$ times acceleration $a$: $F = ma$. But acceleration is the second derivative of position $s(t)$ versus time $t$. So

$$F = m\frac{d^2 s}{dt^2} \quad \text{and} \quad W = Fd.$$

So if acceleration and mass are constant, we have a simple problem or arithmetic.

Example: Lifting a $1.2$ kg book takes a force of $mg$, where $g = 9.8$ m/s$^2$ is the acceleration of gravity. So lifting the book $0.7$ m off the ground takes an effort of

$$W = (1.2 \text{ kg})(9.8 \text{ m/s}^2)(0.7 \text{ m}) = (1.2)(9.8)(0.1) \text{ J},$$

where J is joules ( kg $*$ m$^2$ / s$^2$). Note, 1 kg $*$ m / s$^2$ = N, a Newton.In the US system, we use the pound.

## Non-constant force

If, on the other hand, force is not constant, we need some calculus! Suppose an object is being moved in the positive $x$-direction from $x = 1$ to $x = b$. Let $f(x)$ be the force exerted as a function of position $x$. Then we can approximate $W$ by breaking $[a, b]$ into $n$ subintervals, picking an $x_i$ in each subinterval, and approximating $f(x)$ as $f(x_i)$ over that interval, so that

$$W \approx \sum_{i=1}^{n} f(x_i)\Delta x. \quad \text{Thus } W = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

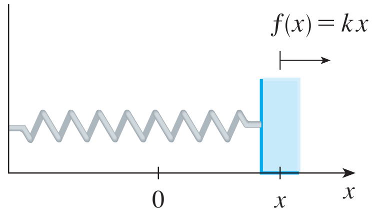
Example: When a particle is moved by a force of $f(x) = x^2 + 2x$ pounds ($x$ in feet), how much work is done by moving is from $x = 1$ to $x = 3$?

$$W = \int_{x=1}^{3} f(x)dx = \int_1^3 x^2 + 2x \; dx = (\tfrac{1}{3}x^3 + x^2)\Big|_1^3 = \tfrac{50}{3}\text{ft-lb}.$$

# Hooke's law

Hooke's law states that the force required to stretch a spring $x$ units beyond equilibrium is proportional to $x$:

$F = kx$, where $k$ is constant, depending on the spring.




0 is equilibrium

Example: Suppose a force of $40$ N is requires to hold a spring 5cm from its equilibrium. How much work is done in stretching is from 5cm to 8 cm from equilibrium?

Change to meters, using 1 cm = 0.01 m. Then using Hooke's law,

$$f(x) = 40 \text{ N } = kx = k0.05 \text{ m}, \quad \text{so } k = 40/.05 = 800.$$

Thus $f(x) = kx = 800x$. So

$$W = \int_{0.05}^{0.08} 800x \, dx = 400x^2 \Big|_{.05}^{.08} = 1.56 \text{ J}.$$