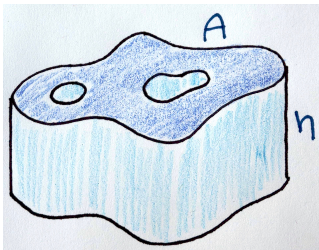
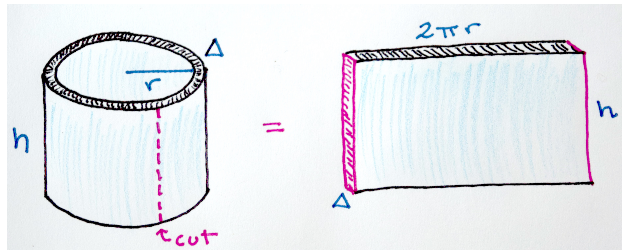
7.3 Volumes by cylindrical shells

Recall from last time that if we have a cylindrical shape with height h and whose face has area A its volume is

$$V(\text{cylinder}) = Ah.$$





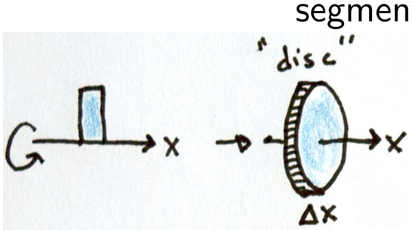
On the other hand, a (circular) cylindrical shell with very small thickness $\Delta = \Delta x$ or Δy, with radius r and height h, has volume

$$V(\text{cylindrical shell}) = 2\pi r h \Delta.$$

(Cut the shell down one side and unfold to get a rectangle; the circumference of the cylinder was $2\pi r$, so the length of the top is the same.)

Rotate a segment that's perpendicular to the axis of rotation,

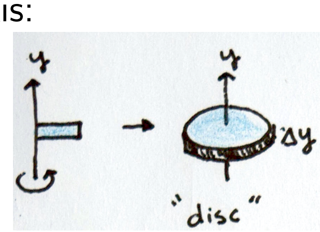
segment touching the axis:



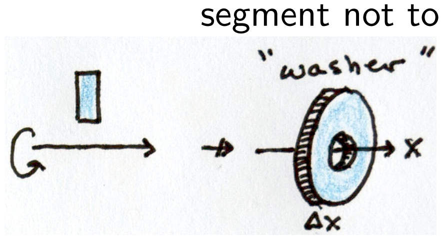
$\Delta V = \pi r^2 \Delta x$

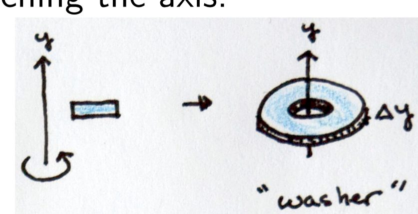
or

$\Delta V = \pi r^2 \Delta y$



segment not touching the axis:

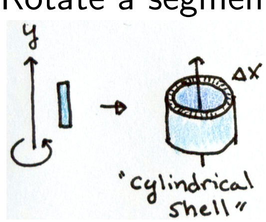




$\Delta V = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\Delta x$

$\Delta V = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\Delta y$

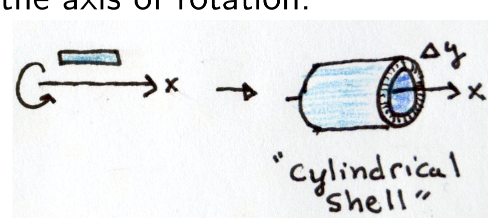
Rotate a segment that's parallel to the axis of rotation:



$\Delta V = 2\pi x h \Delta x$

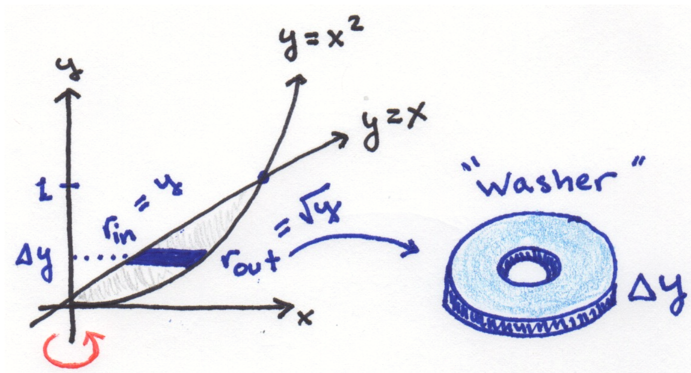
or

$\Delta V = 2\pi y h \Delta y$



Revisiting an example

Suppose we want to rotate the region bounded between $y = x^2$ and $y = x$ around the y-axis. If I take slices perpendicular to the y-axis, I get washers:



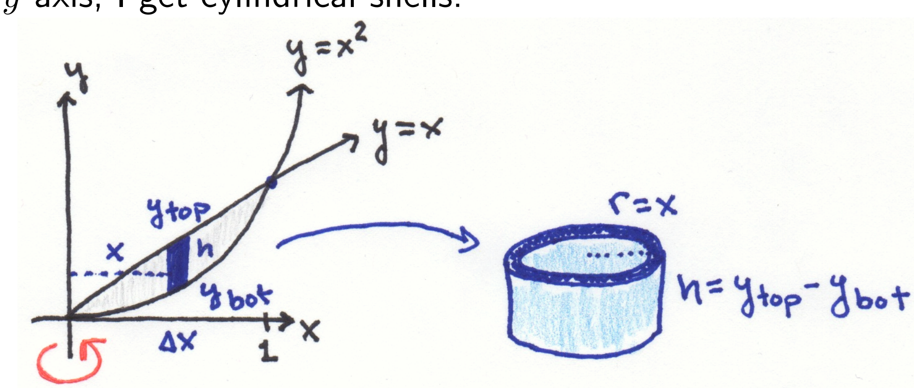
Now the variable is y, the radii are $r_{\text{in}} = x_{\text{left}} = y$ and $r_{\text{out}} = x_{\text{right}} = \sqrt{y}$, so that

$$\Delta V = \pi((\sqrt{y})^2 - y^2)\Delta y,$$

and so $V = \displaystyle\int_0^1 \pi(y - y^2)dy = \pi(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{\pi}{6}$

Revisiting an example

Suppose we want to rotate the region bounded between $y = x^2$ and $y = x$ around the y-axis. If, instead, I take slices parallel to the y-axis, I get cylindrical shells:

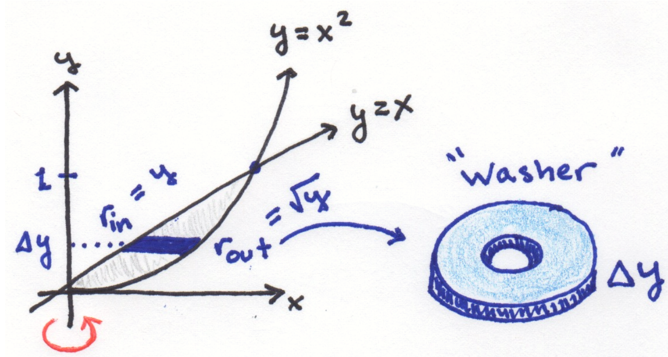


Now the variable is x, so the radius is x, and the height is $h = y_{\text{top}} - y_{\text{bot}} = x - x^2$. Thus

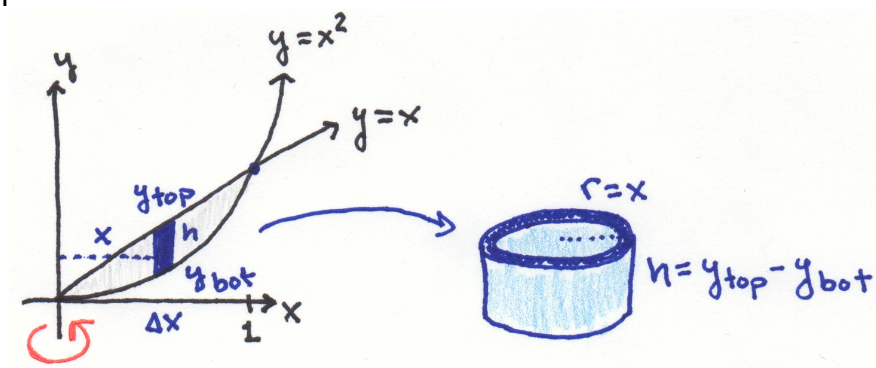
$$\Delta V = 2\pi x(x - x^2)\Delta x,$$

and so $V = \displaystyle\int_0^1 2\pi(x^2 - x^3)dx = 2\pi(\tfrac{1}{3} - \tfrac{1}{4}) = \tfrac{\pi}{6}$

Slice perpendicular to the axis of rotation:



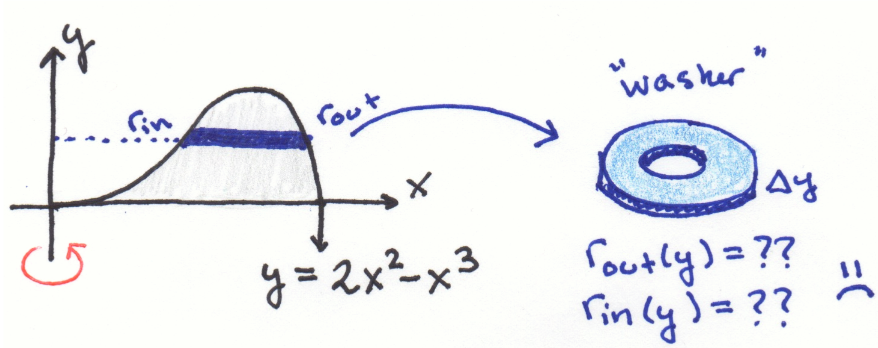
Slice parallel to the axis of rotation:



(Different slices, different integral, same 3D shape, same answer)

Is one method ever "better" than the other?

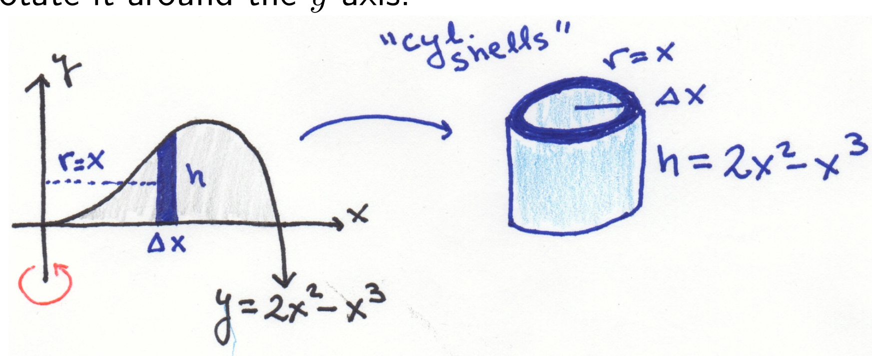
Take the region bounded between $y = 2x^2 - x^3$ and the x-axis, and rotate it around the y-axis:



If I try to do washers (like last time), I run into the problem of inverting $f(x) = 2x^2 - x^3$, separately over the intervals $[0, 4/3]$ and $[4/3, 2]$. Yuck!

Is one method ever "better" than the other?

Take the region bounded between $y = 2x^2 - x^3$ and the x-axis, and rotate it around the y-axis:



Instead, I want to use cylindrical shells! Now, the radius is x, the thickness is Δx, and the height is $h = y_{\text{top}} - y_{\text{bot}} = (2x^2 - x^3) - 0$. So $\Delta V = 2\pi x(2x^2 - x^3)\Delta x$. The curve intersects the x-axis at when $2x^2 - x^3 = 0$, i.e. $x = 0$ and $x = 2$. So

$$V = \int_0^2 2\pi x(2x^2 - x^3)\, dx = 2\pi \int_0^2 2x^3 - x^4\, dx = 2\pi \left(\tfrac{2}{4}x^4 - \tfrac{1}{5}x^5\right)\Big|_0^2$$

$$= 2\pi\left(\tfrac{1}{2}2^4 - \tfrac{1}{5}2^5\right) - 0.$$

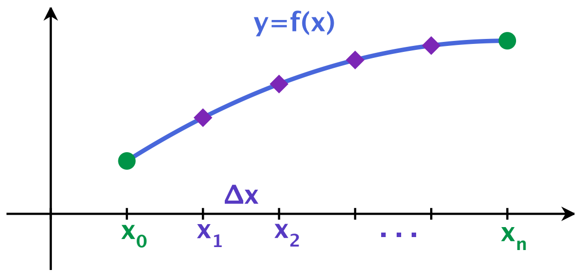
You try:

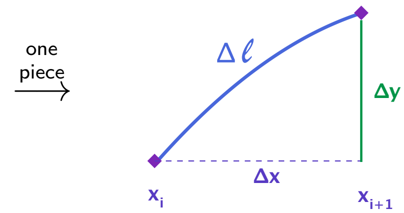
For each of the following, (a) sketch a picture of the region, (b) pick a method – discs/washers or cylindrical shells, and (c) compute the volume of the described solid.

1. Rotate $\mathcal{R}$ around the y-axis, where $\mathcal{R}$ is the region bounded between $y = -(x-1)^2 + 2$ and the x-axis.
2. Rotate $\mathcal{R}$ around the x-axis, where $\mathcal{R}$ is the region bounded between $y = -(x-1)^2 + 2$ and the x-axis.
3. Rotate $\mathcal{R}$ around the y-axis, where $\mathcal{R}$ is the region bounded below $y = \frac{1}{2}x^2$, above the x-axis, and below the line $y = 2x - 2$.

7.4: Arc length

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



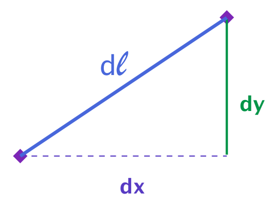


Slice!

Let n go to ∞

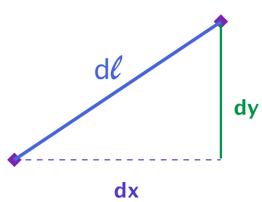
$\downdownarrows$

$$\ell = \lim_{n \to \infty} \sum_{i=1}^{n} (\Delta s)_i = \int_{x=a}^{x=b} d\ell$$



$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



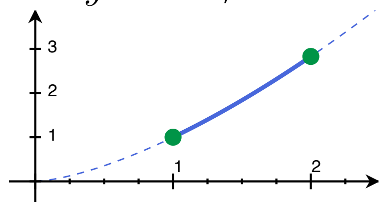
Remember, $y = f(x)$.

$$\begin{aligned} d\ell = \sqrt{dx^2 + dy^2} &= \sqrt{dx^2 + dy^2}\,\frac{dx}{dx} \\ &= \sqrt{\frac{dx^2 + dy^2}{dx^2}}\,dx = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\,dx \\ &= \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1 + (f'(x))^2}\,dx \end{aligned}$$

So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + (f'(x))^2}\,dx}$$

Arc length function

Find the length of the arc $y = x^{3/2}$, from $x = 1$ to $x = 2$.



$$f(x) = x^{3/2} \quad \Longrightarrow \quad f'(x) = \frac{3}{2}x^{1/2}$$

So

$$1 + (f'(x))^2 = 1 + \left(\frac{3}{2}x^{1/2}\right)^2 = 1 + \frac{9}{4}x$$

So

$$\ell = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx = \int_1^2 \left(1 + \frac{9}{4}x\right)^{1/2}\, dx$$

$$= \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_{x=1}^{2} = \boxed{\tfrac{8}{27}\left(\left(1 + \tfrac{9}{4}\cdot 2\right)^{3/2} - \left(1 + \tfrac{9}{4}\right)^{3/2}\right)}$$

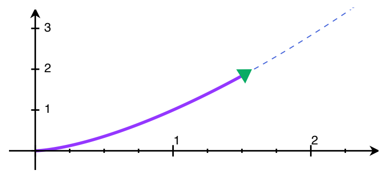
You try:

Set up (but do not integrate) the integrals which compute the length of the following functions. Notice that most of the time, the resulting integral is "hard" (not elementary).

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1

Arc length function

Suppose, instead, I have a particle traveling along the same curve, $y = x^{3/2}$, starting at $x = 0$, and traveling in the positive direction. How far has the particle traveled as a function of x?



We saw that the arc length for this function over the interval $[a, b]$ is $\ell = \int_a^b \sqrt{1 + \frac{9}{4}x}\ dx$. But I want my interval to be $[0, x]$, so I need to change my variable inside the integral:

$$\ell(x) = \int_0^x \sqrt{1 + \frac{9}{4}t}\ dt = \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(1 + \frac{9}{4}t\right)^{3/2}\Bigg|_{t=0}^{x}$$

$$= \boxed{\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(\left(1 + \frac{9}{4}x\right)^{3/2} - 1\right)}$$

You try

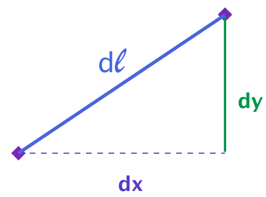
Compute the distance traveled, $\ell(x)$ along the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$, starting at the point $(1, 1/2)$ and traveling in the positive x-direction.

Hint:

1. Compute dy/dx.
2. Compute $1 + (dy/dx)^2$ and simplify. Notice that this factors as a perfect square.
3. Simplify $g(x) = \sqrt{1 + (dy/dx)^2}$.
4. Change variables and compute $\int_1^x g(t)\ dt$.

Arc length of functions $x = f(y)$.

At the beginning, we had $d\ell = \sqrt{dx^2 + dy^2}$ and multiplied both sides by dx/dx.



Now, suppose we want to know the length of a curve $x = f(y)$ for $a \leq y \leq b$. So now, multiply both sides by dy/dy!

$$d\ell = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + dy^2}\,\frac{dy}{dy} = \sqrt{\frac{dx^2 + dy^2}{dy^2}}\,dy$$

$$= \sqrt{\frac{dx^2}{dy^2} + \frac{dy^2}{dy^2}}\,dy = \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy$$

So $$\boxed{\ell = \int_{y=a}^{b} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy}$$

You try

Compute the arclength of $y = \arccos(e^x)$ for $\ln(1/2) \leq x \leq 0$.
Hint:

1. Try setting it up in terms of x first.
2. Realize (1) is terrible. Solve $y = \arccos(e^x)$ for x instead. Compute the new bounds for y.
3. Calculate the arclength in terms of y. You may need some trig integral stuff.