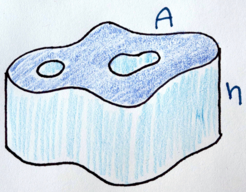
7.3 Volumes by cylindrical shells

Recall from last time that if we have a cylindrical shape with height h and whose face has area A its volume is

$$V(\text{cylinder}) = Ah.$$



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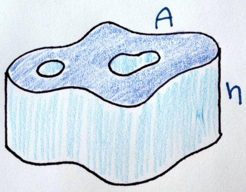
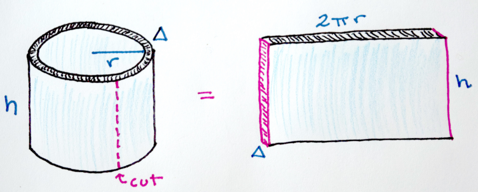


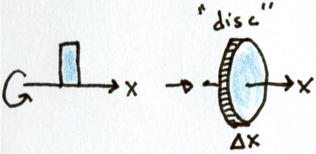


On the other hand, a (circular) cylindrical shell with very small thickness $\Delta = \Delta x$ or Δy, with radius r and height h, has volume

$$V(\text{cylindrical shell}) = 2\pi r h \Delta.$$

(Cut the shell down one side and unfold to get a rectangle; the circumference of the cylinder was $2\pi r$, so the length of the top is the same.)

Rotate a segment that's perpendicular to the axis of rotation, segment touching the axis:



$$\boxed{\Delta V = \pi r^2 \Delta x}$$

or

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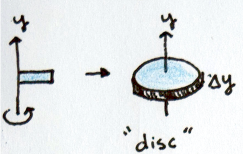

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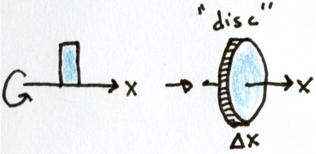

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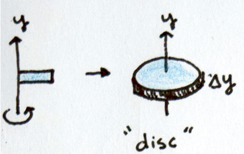
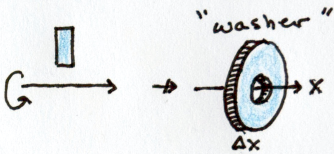
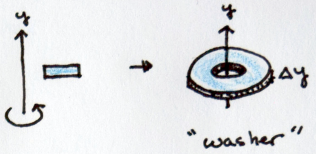


segment not touching the axis:





$$\boxed{\Delta V = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\Delta x}$$

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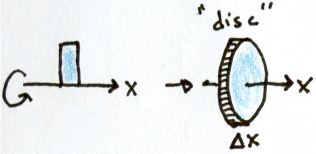

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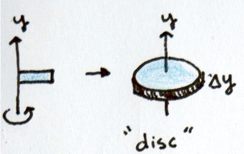

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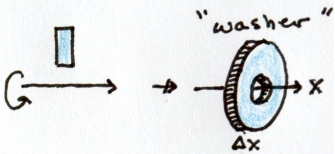

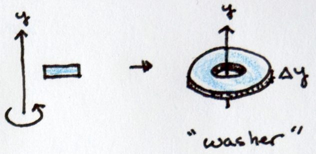

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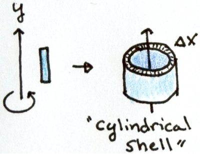
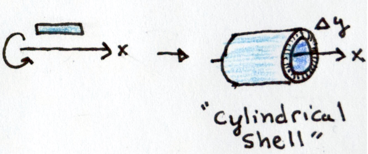


$$\Delta V = 2\pi x h \Delta x$$

or

$$\Delta V = 2\pi y h \Delta y$$



Revisiting an example

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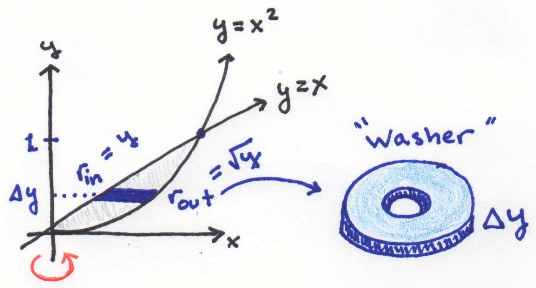

Now the variable is y, the radii are $r_{\text{in}} = x_{\text{left}} = y$ and $r_{\text{out}} = x_{\text{right}} = \sqrt{y}$, so that

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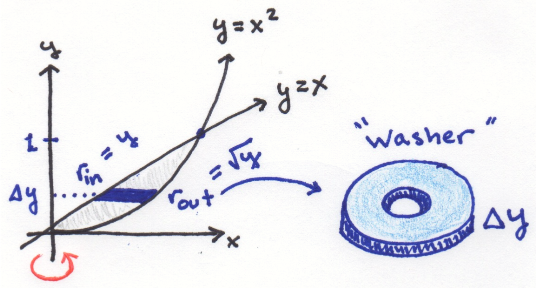

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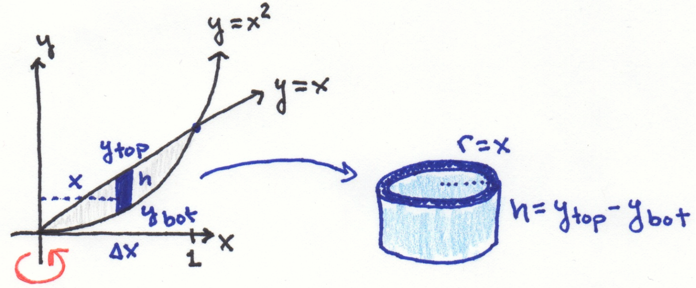

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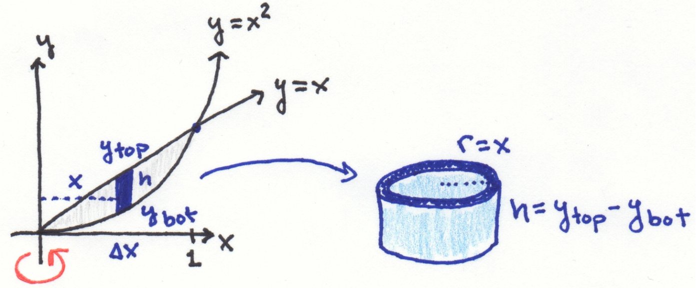

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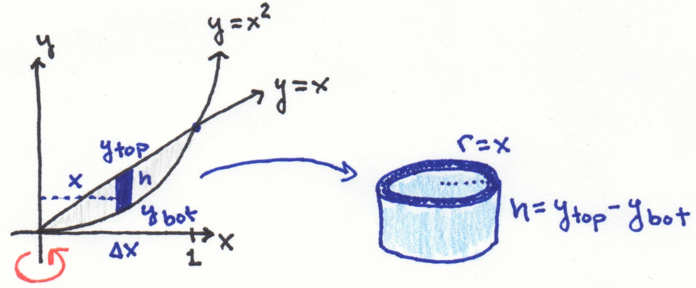

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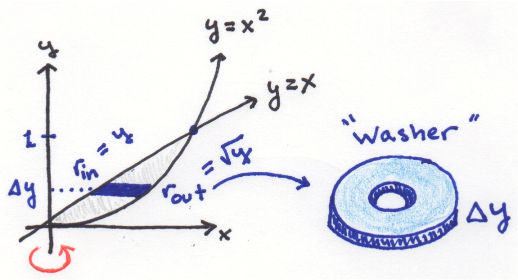
Slice perpendicular to the axis of rotation:



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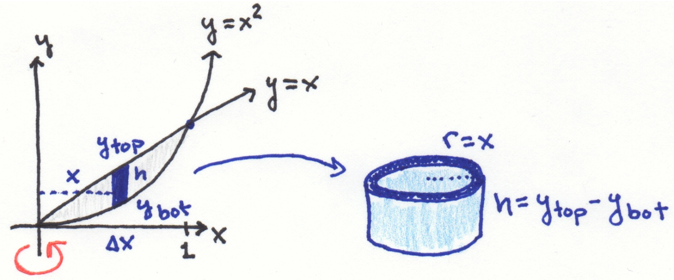

(Different slices, different integral, same 3D shape, same answer)

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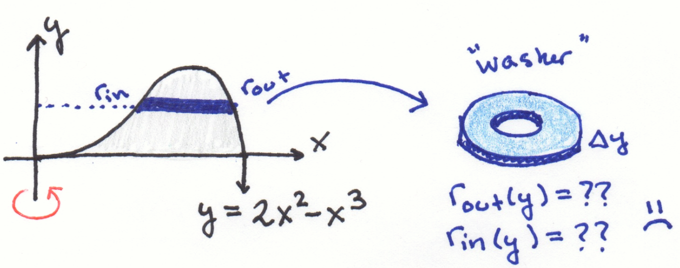

If I try to do washers (like last time), I run into the problem of inverting $f(x) = 2x^2 - x^3$, separately over the intervals $[0, 4/3]$ and $[4/3, 2]$. Yuck!

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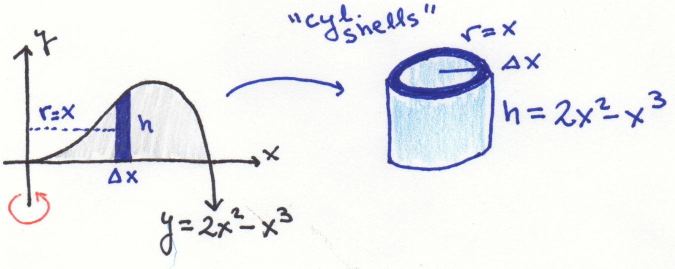

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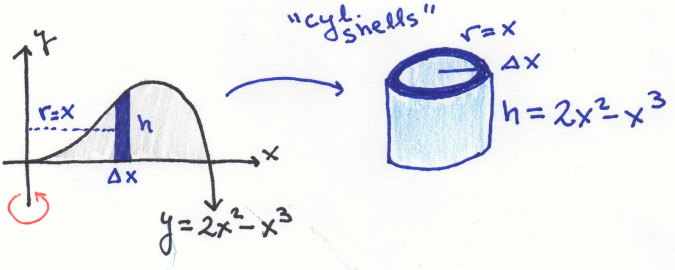

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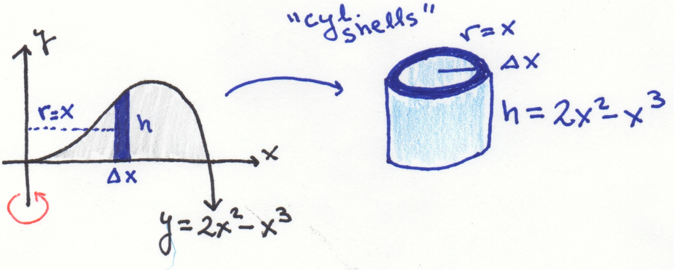

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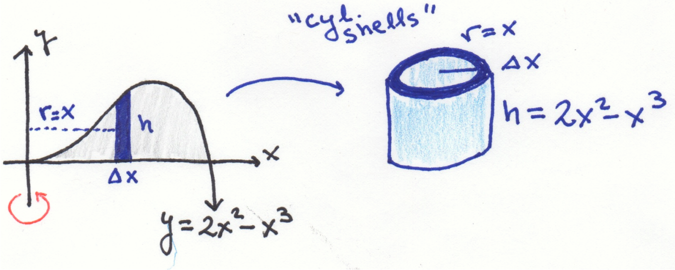

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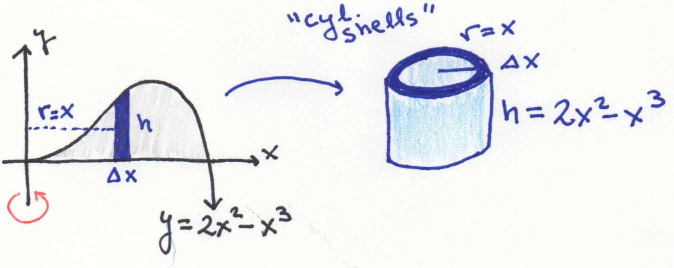

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$$V = \int_0^2 2\pi x(2x^2 - x^3)\, dx$$

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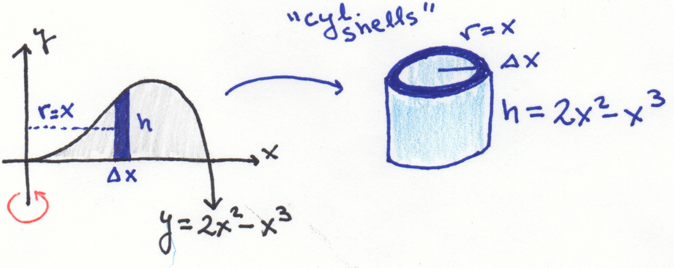

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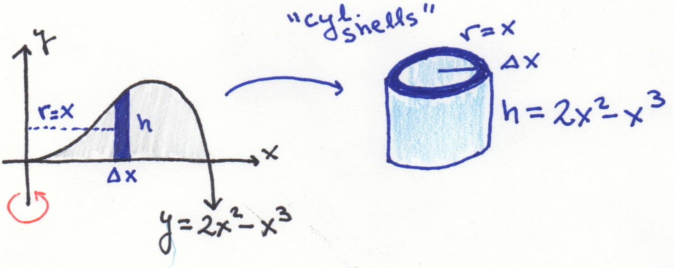

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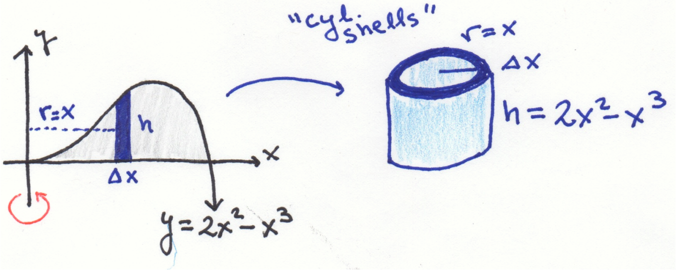

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$$= 2\pi \left(\tfrac{1}{2}2^4 - \tfrac{1}{5}2^5\right) - 0.$$

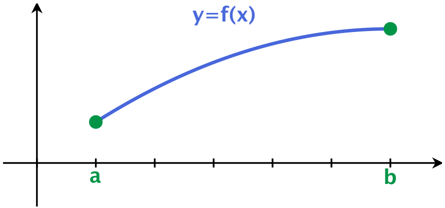
You try:

For each of the following, (a) sketch a picture of the region, (b) pick a method – discs/washers or cylindrical shells, and (c) compute the volume of the described solid.

1. Rotate $\mathcal{R}$ around the y-axis, where $\mathcal{R}$ is the region bounded between $y = -(x-1)^2 + 2$ and the x-axis.
2. Rotate $\mathcal{R}$ around the x-axis, where $\mathcal{R}$ is the region bounded between $y = -(x-1)^2 + 2$ and the x-axis.
3. Rotate $\mathcal{R}$ around the y-axis, where $\mathcal{R}$ is the region bounded below $y = \frac{1}{2}x^2$, above the x-axis, and below the line $y = 2x - 2$.

7.4: Arc length

Suppose you want to know what the length of a curve $y = f(x)$ is from the point $(a, f(a))$ to the point $(b, f(b))$:



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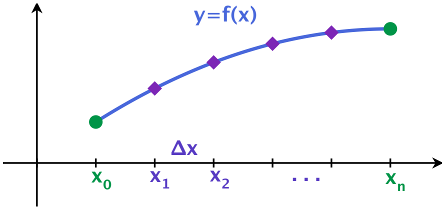

Slice!

$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} \text{(little length)}_i$$

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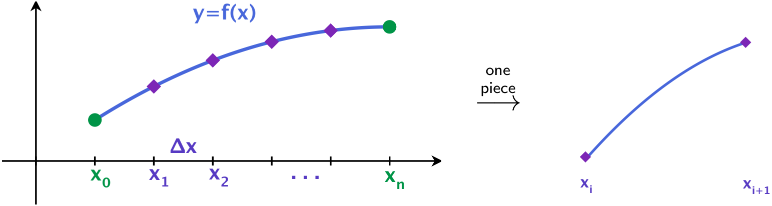

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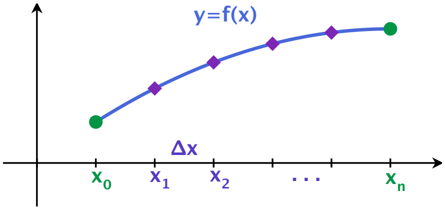

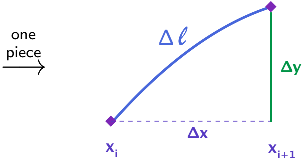

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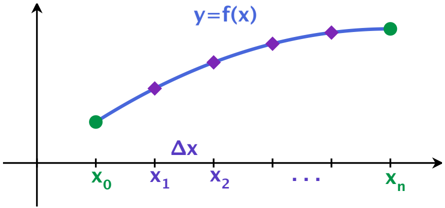
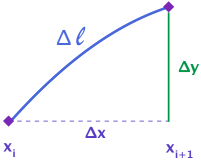


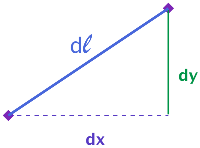
Slice!

one piece $\longrightarrow$



Let n go to ∞

$\Downarrow$



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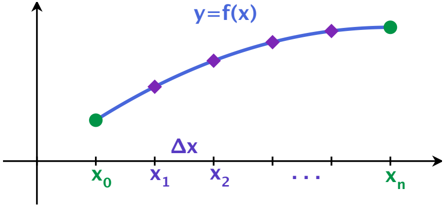

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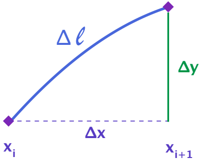
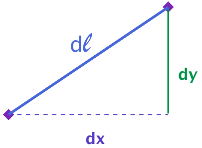


Slice!

Let n go to ∞

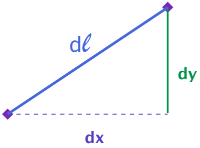
$\downdownarrows$



$$\ell = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta s)_i = \int_{x=a}^{x=b} d\ell$$

$$\boxed{d\ell = \sqrt{dx^2 + dy^2}}$$

Manipulating into something we can actually calculate...



Remember, $y = f(x)$.

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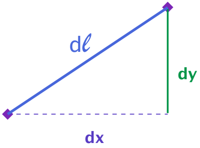

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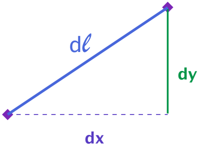

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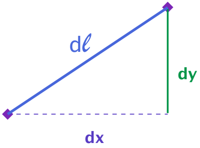

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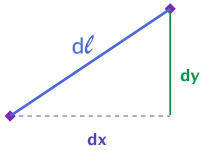

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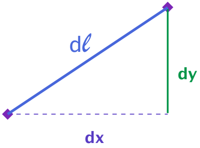

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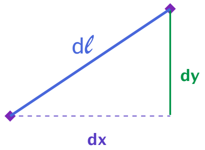

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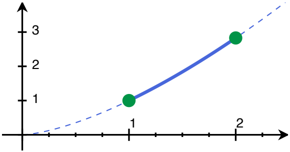
So $$\boxed{\ell = \int_{x=a}^{b} \sqrt{1 + \left(f'(x)\right)^2}\ dx}$$

Arc length function

Find the length of the arc $y = x^{3/2}$, from $x = 1$ to $x = 2$.



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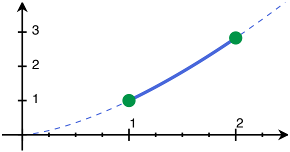

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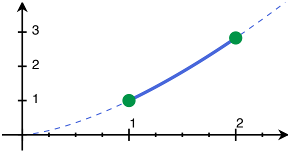

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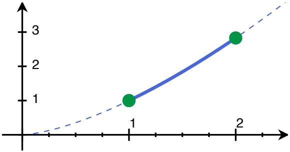

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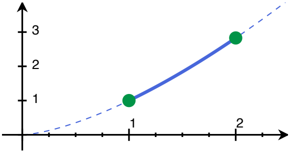

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$$= \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_{x=1}^{2} = \boxed{\frac{8}{27}\left(\left(1 + \tfrac{9}{4}\cdot 2\right)^{3/2} - \left(1 + \tfrac{9}{4}\right)^{3/2}\right)}$$

You try:

Set up (but do not integrate) the integrals which compute the length of the following functions. Notice that most of the time, the resulting integral is “hard” (not elementary).

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
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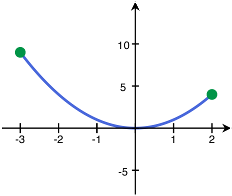
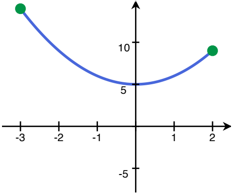
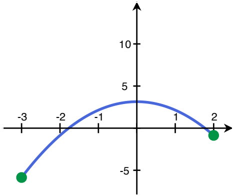
Notice that the first three have the same arc length!

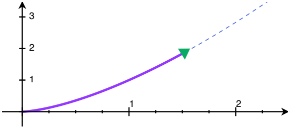






Arc length function

Suppose, instead, I have a particle traveling along the same curve, $y = x^{3/2}$, starting at $x = 0$, and traveling in the positive direction. How far has the particle traveled as a function of x?



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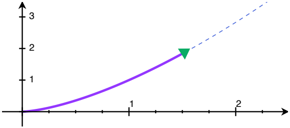

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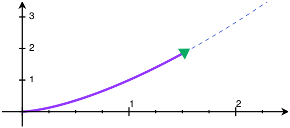

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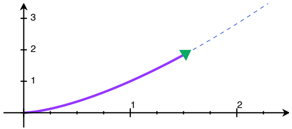

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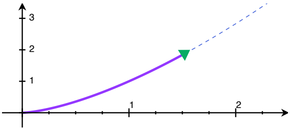

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$$= \boxed{\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(\left(1 + \frac{9}{4}x\right)^{3/2} - 1\right)}$$

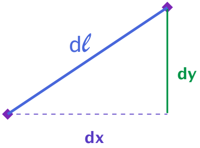
You try

Compute the distance traveled, $\ell(x)$ along the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$, starting at the point $(1, 1/2)$ and traveling in the positive x-direction.
Hint:

1. Compute dy/dx.
2. Compute $1 + (dy/dx)^2$ and simplify. Notice that this factors as a perfect square.
3. Simplify $g(x) = \sqrt{1 + (dy/dx)^2}$.
4. Change variables and compute $\int_1^x g(t)\ dt$.

Arc length of functions $x = f(y)$.

At the beginning, we had $d\ell = \sqrt{dx^2 + dy^2}$ and multiplied both sides by dx/dx.



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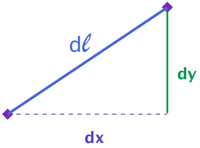

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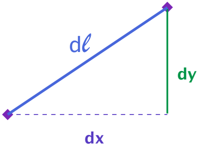

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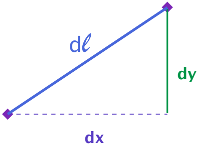

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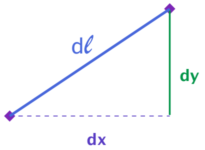

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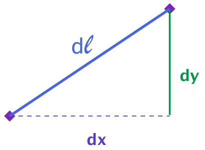

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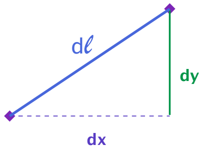

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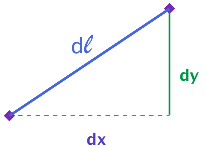

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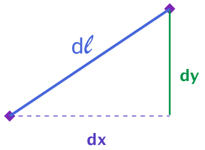

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So $$\boxed{\ell = \int_{y=a}^{b} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy}$$

You try

Compute the arclength of $y = \arccos(e^x)$ for $\ln(1/2) \le x \le 0$.
Hint:

1. Try setting it up in terms of x first.
2. Realize (1) is terrible. Solve $y = \arccos(e^x)$ for x instead. Compute the new bounds for y.
3. Calculate the arclength in terms of y. You may need some trig integral stuff.