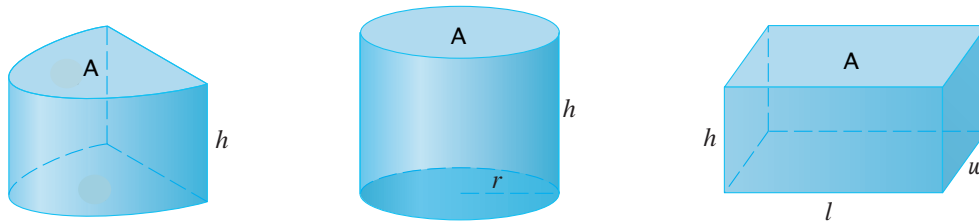


## 7.2 Volumes

A **right cylinder** (not necessarily circular) is a 3-dimensional solid made by starting with a 2-dimensional shape (like a half-circle, a circle, a rectangle, or whatever), filling it in, and then translating it straight up for a height  $h$ .



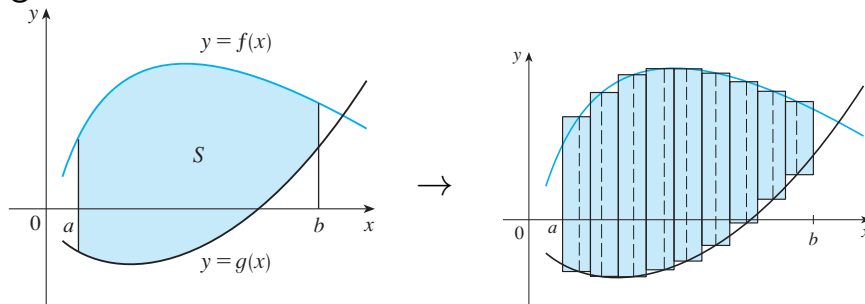
The volume of such a shape is the area  $A$  of the base times the height  $h$ :

$$V = Ah.$$

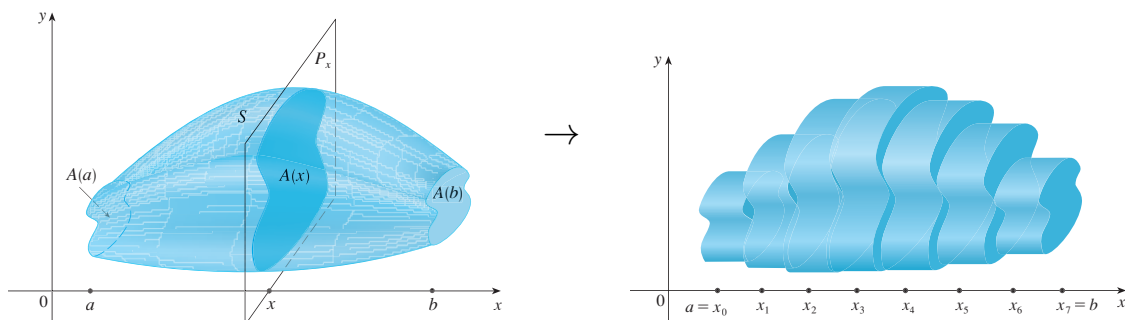
For the circular cylinder, the base has area  $A = \pi r^2$ , so  $V = \pi r^2 h$ . The rectangular cylinder has base of area  $A = lw$ , so  $V = lwh$ .

## Non-cylindrical solids

Just like we approximated 2-dimensional areas as a bunch of rectangles:

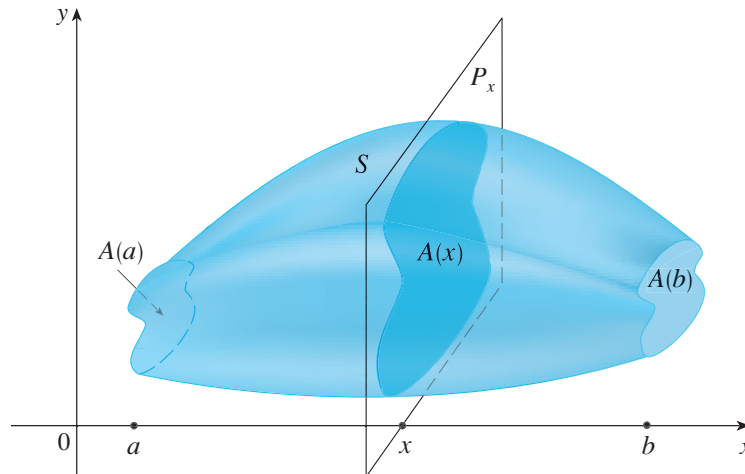


we can approximate 3-dimensional volumes as a bunch of cylinders:



## Non-cylindrical solids

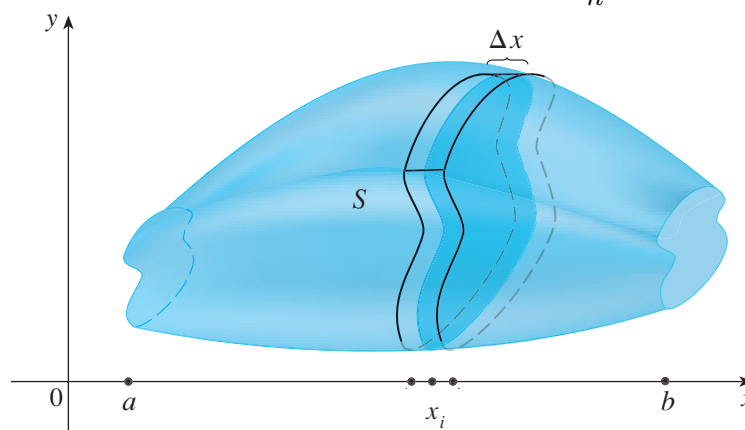
We start with some 3D solid  $S$ , and take cross-sections of it perpendicular to one of the axes (say  $x$ ):



Here, the shape goes out to the left as far as  $x = a$  and to the right as far as  $x = b$ . Let  $A(x)$  be the area of the cross-section of the shape at  $x$ . So the far left cross-section has area  $A(a)$ , and the far right cross-section has area  $A(b)$ .

## Non-cylindrical solids

Breaking the interval  $[a, b]$  into  $n$  pieces (just like before), we pick an  $x_i$  in each interval. Then we approximate the volume of the shape “near by” each  $x_i$  by a cylinder whose face is the shape of the cross-section, and whose height is  $\Delta x = \frac{b-a}{n}$ :



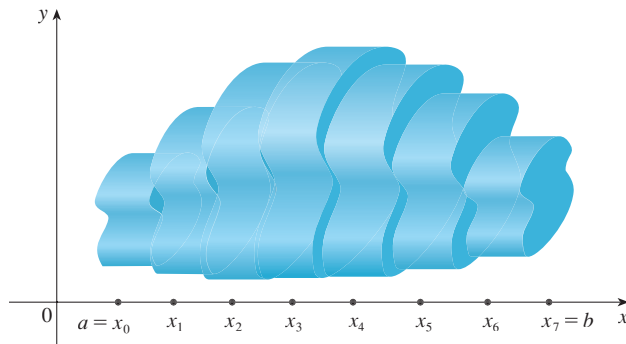
Since the area of the cross-section is  $A(x)$ , the volume of that (very short) cylinder is

$$\Delta V_i = A(x_i)\Delta x.$$

## Non-cylindrical solids

The volume of each cylinder is  $\Delta V_i = A(x_i)\Delta x$ . Summing up, we approximate the volume by

$$V \approx \sum_{i=1}^n A(x_i)\Delta x.$$

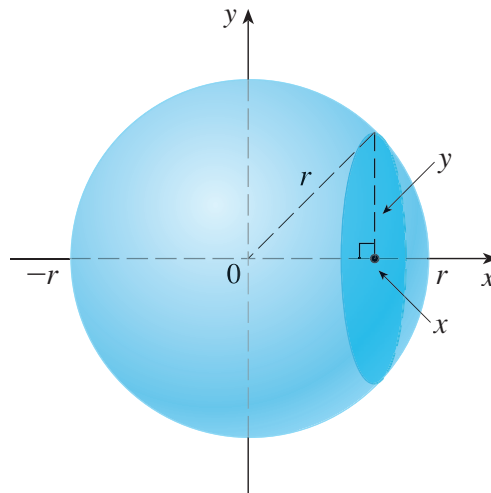


As before, the more subintervals we have, the better the approximation. So the exact volume is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i)\Delta x = \int_a^b A(x)dx \quad \text{by definition.}$$

## Example: Calculating the volume of a sphere

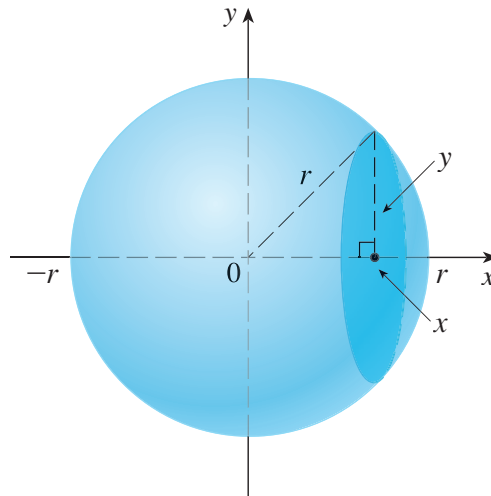
Place a sphere of radius  $r$  with its center at the origin:



The cross-sections perpendicular to the  $x$ -axis are circles. For a fixed  $x$ , the cross-section's area depends on the radius  $\rho(x)$ . So what is the radius  $\rho(x)$  of the corresponding circle??

## Example: Calculating the volume of a sphere

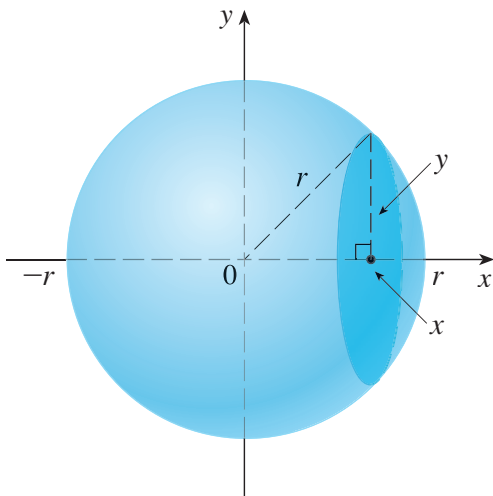
Place a sphere of radius  $r$  with its center at the origin:



We have the triangle pictured, whose base is  $x$  (since that's how far out we are), whose hypotenuse is  $r$  (the radius of the whole sphere), and whose height is  $y = \rho(x)$ . So  $\rho^2(x) = r^2 - x^2$ . Thus

$$A(x) = \pi \rho^2(x) = \pi(r^2 - x^2).$$

## Example: Calculating the volume of a sphere



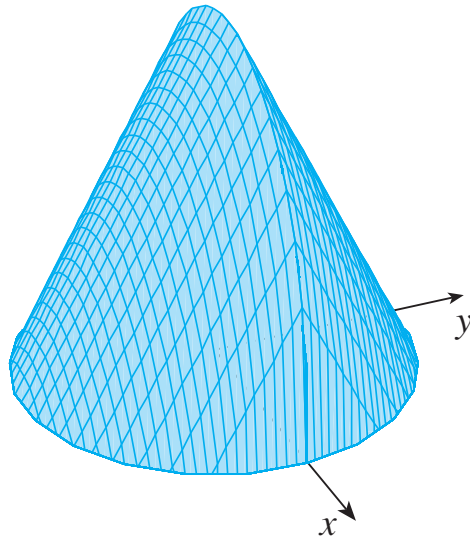
$$A(x) = \pi(r^2 - x^2) \quad (r \text{ is constant})$$

So using  $V = \int_a^b A(x) dx$ , we have

$$\begin{aligned} V &= \int_{-r}^r A(x) dx = \int_{-r}^r \pi(r^2 - x^2) dx = \int_{-r}^r \pi r^2 - \pi x^2 dx \\ &= \pi r^2 x - \frac{\pi}{3} x^3 \Big|_{-r}^r = \pi r^3 - \frac{\pi}{3} r^3 - (-\pi r^3 + \frac{\pi}{3} r^3) = \boxed{\frac{4}{3} \pi r^3} !! \end{aligned}$$

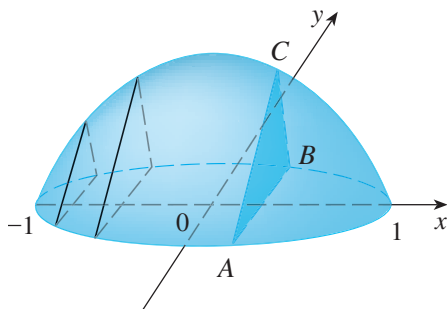


Suppose we have a 3D shape  $S$  that can be described as follows:  
 (\*)  $S$  has circular base of radius 1, and (\*) parallel cross-sections perpendicular to the base are equilateral triangles.  
 $S$  :

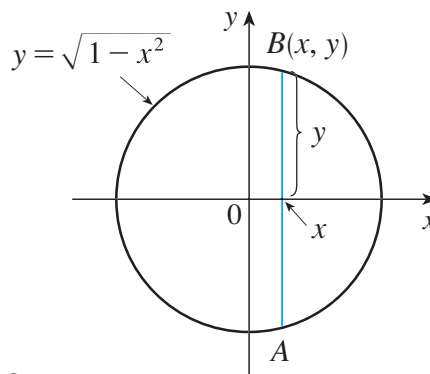


Suppose we have a 3D shape  $S$  that can be described as follows:  
 (\*)  $S$  has circular base of radius 1, and (\*) parallel cross-sections perpendicular to the base are equilateral triangles.

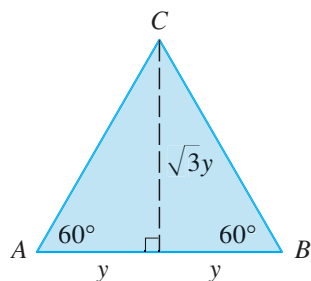
$S$  with cross-sections :



Base of  $S$  :



Slices: height  $\Delta x$  and triangular face



$$A(x) = \frac{1}{2}y * \sqrt{3}y = \frac{\sqrt{3}}{2}y^2$$

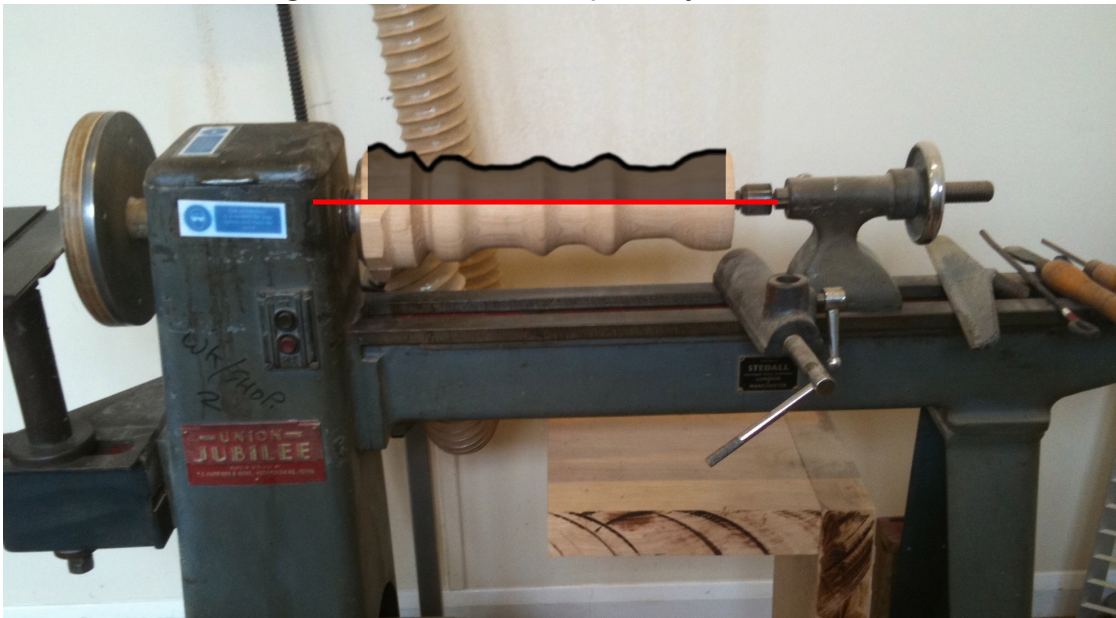
$$= \frac{\sqrt{3}}{2}(1 - x^2)$$

You try: What is  $V$ ?

## Volumes of revolution

Now say we have a volume that can be described as a 2-dimensional shape rotated around an axis.

Think of working with a lathe or a pottery wheel:

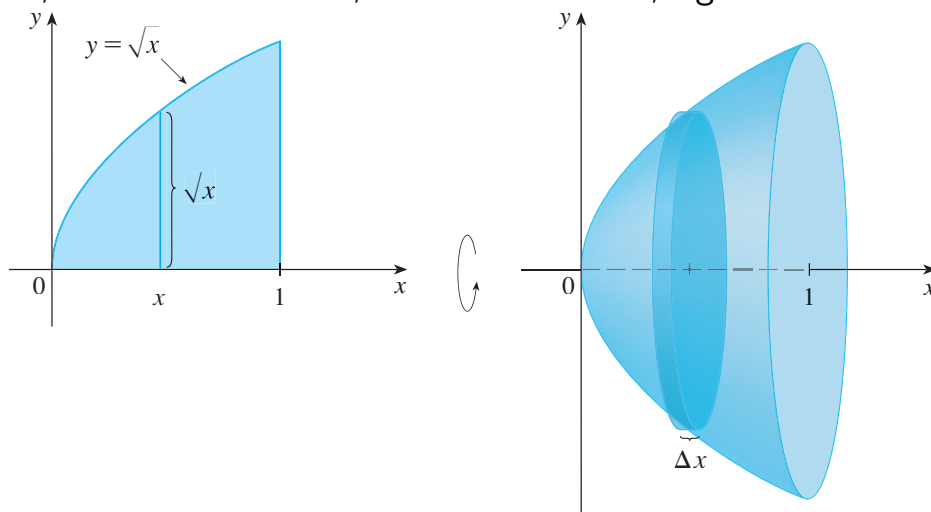


How much wood is left?

## Volumes of revolution

Now say we have a volume that can be described as a 2-dimensional shape rotated around an axis.

For example, if I rotate the area bounded between  $y = \sqrt{x}$ , the  $x$ -axis, and the line  $x = 1$ , around the  $x$ -axis, I get

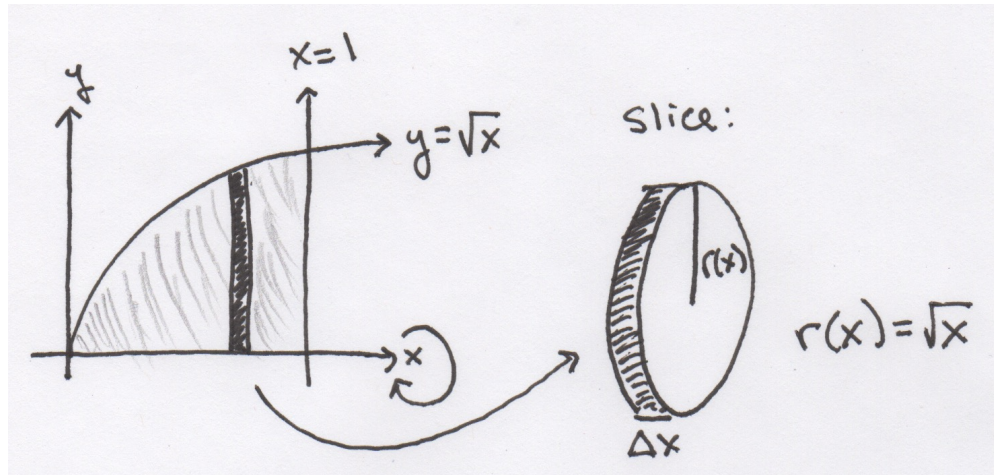


We do the same thing as before, adding up volumes of slices.

**Slices:** Circular cylinders of height  $\Delta x$  and radius  $\sqrt{x}$ .

What is  $A(x)$ ? What is  $V$ ?

Answer:



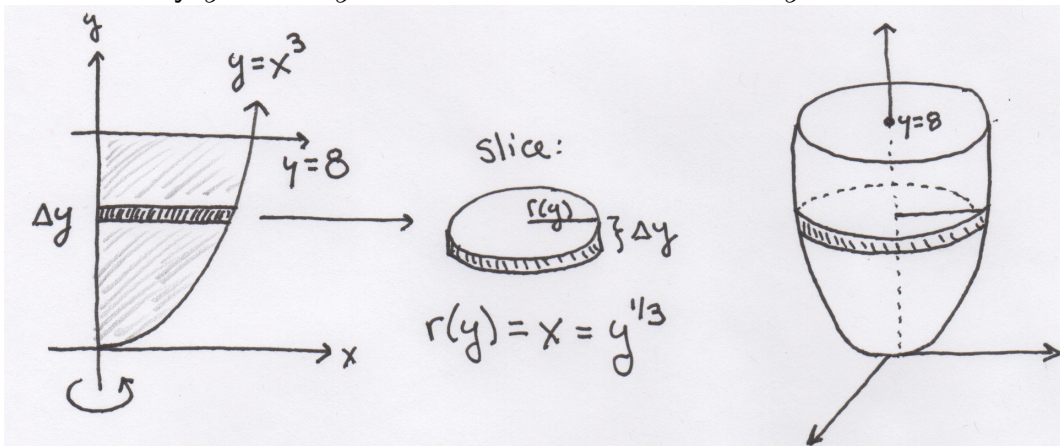
$$A(x) = \pi r^2(x) = \pi(\sqrt{x})^2 = \pi x$$

So

$$V = \int_0^1 A(x) dx = \int_0^1 \pi x dx = \frac{\pi}{3} x^3 \Big|_{x=0}^1 = \frac{\pi}{3}.$$

### Example

Find the volume of the solid obtained by rotating the region bounded by  $y = x^3$ ,  $y = 8$ , and  $x = 0$  about the  $y$ -axis.



The issue here is that the circular cross-sections are horizontal now!

**Slices:** Circular cylinders, with height  $\Delta y$  and radius  $r(y) = \sqrt[3]{y}$ . So

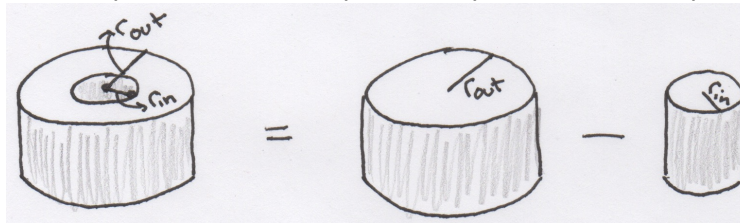
$$A(y) = \pi r^2(y) = \pi y^{2/3}.$$

And thus

$$V = \int_0^8 A(y) dy = \int_0^8 \pi y^{2/3} dy = \frac{3\pi}{5} y^{5/3} \Big|_0^8 = \frac{3\pi}{5} 2^5.$$

## Washers: Volumes that are not convex

Notice that the volume of a hollowed out circular cylinder is  
 $\text{Vol}(\text{Outer cylinder}) - \text{Vol}(\text{Inner cylinder})$



We call these **washers**. If the radius of the outer circular cylinder is  $r_{\text{out}}$  and the radius of the inner circular cylinder is  $r_{\text{in}}$ , then the volume of the washer is

$$V(\text{washer}) = \pi r_{\text{out}}^2 h - \pi r_{\text{in}}^2 h = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)h.$$

Note this is the same as before, except that the face of the cylinder is not convex anymore. But the area of the face is the area of the big circle minus the area of the small circle:

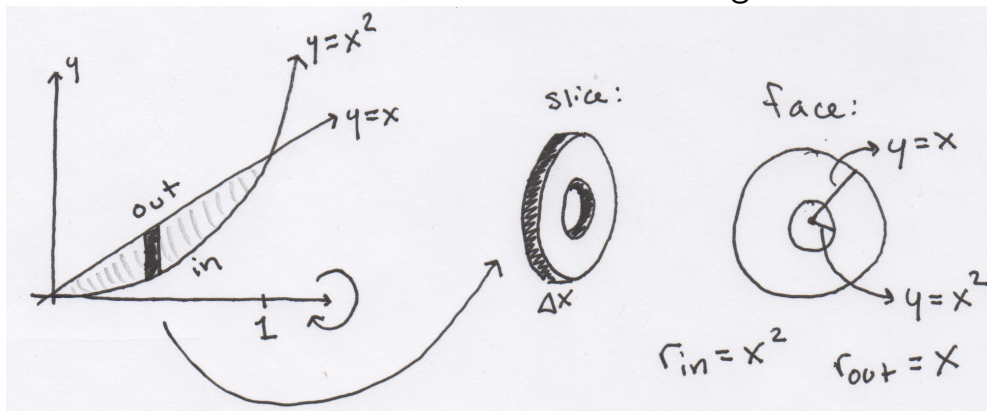
$$A = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2,$$

so that  $V = hA$  give the same answer as above!

## Washers: Volumes that are not convex

$$V(\text{washer}) = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)h.$$

**Example:** Take the region bounded by  $y = x$  and  $y = x^2$ , and rotate it around the  $x$ -axis. What is the resulting volume?



**Slice:** Washer, with height  $\Delta x$  and area  $A = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)$ . Since

$$r_{\text{out}} = x \text{ and } r_{\text{in}} = x^2, \quad A(x) = \pi(x^2 - x^4).$$

$$\text{So } V = \int_0^1 A(x) dx = \int_0^1 \pi(x^2 - x^4) dx = \pi \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left( \frac{1}{3} - \frac{1}{5} \right).$$

## You try

Take the region bounded by  $y = 4$ ,  $y = 4x^2$ , and  $x = 0$ .

1. Draw the region  $y = 4$ ,  $y = 4x^2$ , and  $x = 0$ . What are the end-points of this region in terms of  $x$ ? in terms of  $y$ ?
2. Rotating around the  $x$ -axis:
  - (a) When you rotate this shape around the  $x$ -axis, what is the shape of the slices?
  - (b) For each slice, what the height? What is the variable? Do I want  $A(x)$  or  $A(y)$ ?
  - (c) What is  $A(x)$  or  $A(y)$  (whichever you chose above)?
  - (d) What is the volume?
3. Rotating around the  $y$ -axis:
  - (a) When you rotate this shape around the  $y$ -axis, what is the shape of the slices?
  - (b) For each slice, what the height? What is the variable? Do I want  $A(x)$  or  $A(y)$ ?
  - (c) What is  $A(x)$  or  $A(y)$  (whichever you chose above)?
  - (d) What is the volume?



## General strategy for calculating volumes

1. What are your slices? So far, this is some sort of cylinder with face of area  $A$  and very small height.



2. What is your variable? This should be the thickness of your slice ( $\Delta x$ ,  $\Delta y$ ,  $\Delta z$ ).
3. What are the endpoints with respect to the variable?
4. What is the area of the face of the slice (in terms of the variable)?

Integrate the area in part 4 versus the variable in part 2, between the endpoints in part 3.

**Tips:** Draw lots of pictures, labeling everything. Write each part explicitly. Volume should always be positive.