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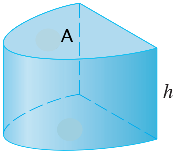

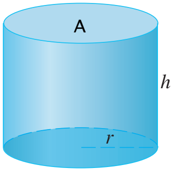

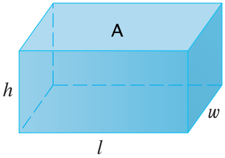

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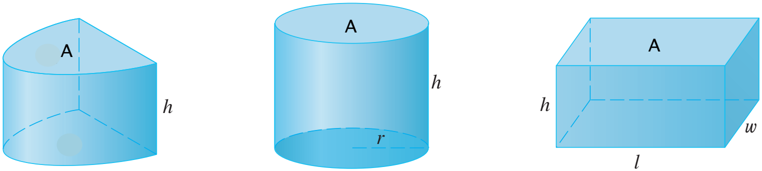

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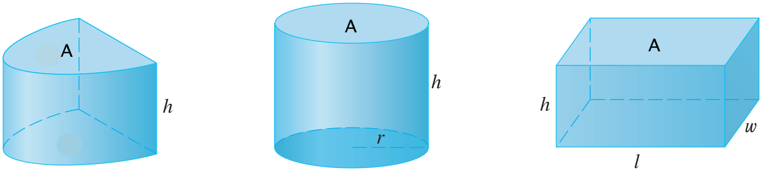

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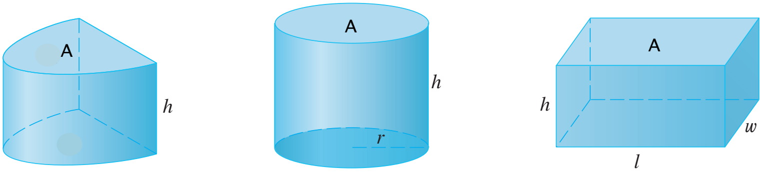

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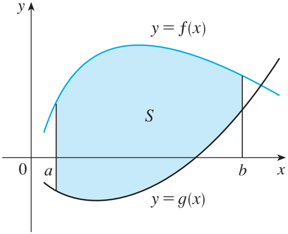
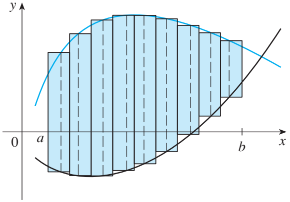
For the circular cylinder, the base has area $A = \pi r^2$, so $V = \pi r^2 h$. The rectangular cylinder has base of area $A = lw$, so $V = lwh$.

Non-cylindrical solids

Just like we approximated 2-dimensional areas as a bunch of rectangles:



$\rightarrow$



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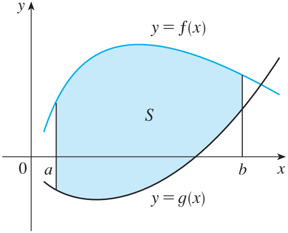

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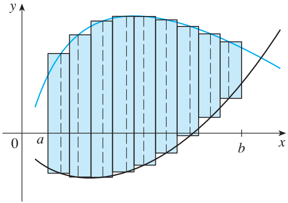
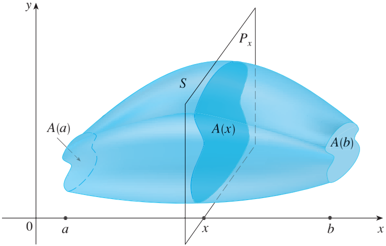


we can approximate 3-dimensional volumes as a bunch of cylinders:



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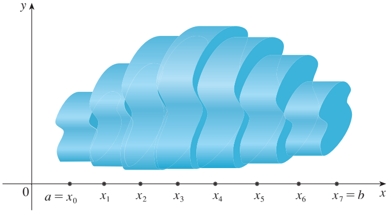

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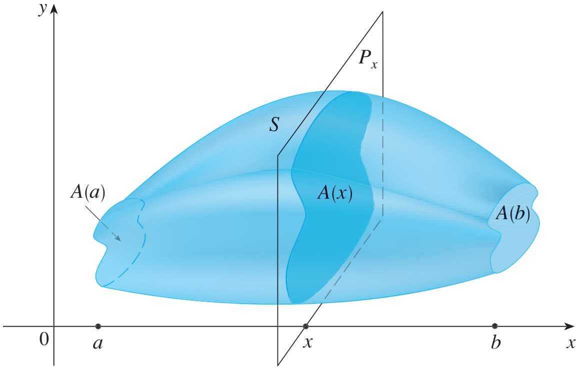

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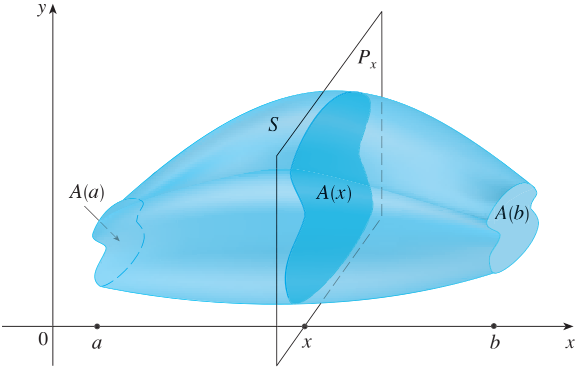

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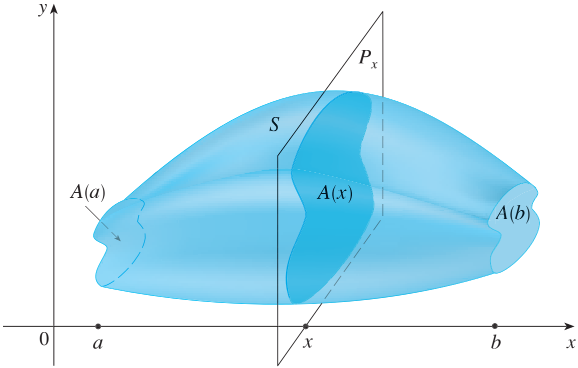

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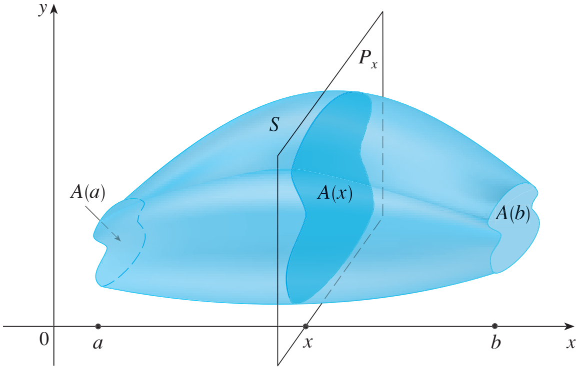
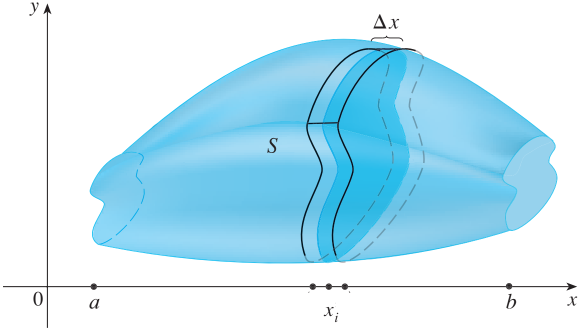


Here, the shape goes out to the left as far as $x = a$ and to the right as far as $x = b$. Let $A(x)$ be the area of the cross-section of the shape at x. So the far left cross-section has area $A(a)$, and the far right cross-section has area $A(b)$.

Non-cylindrical solids

Breaking the interval $[a, b]$ into n pieces (just like before), we pick an x_i in each interval. Then we approximate the volume of the shape "near by" each x_i by a cylinder whose face is the shape of the cross-section, and whose height is $\Delta x = \frac{b-a}{n}$:



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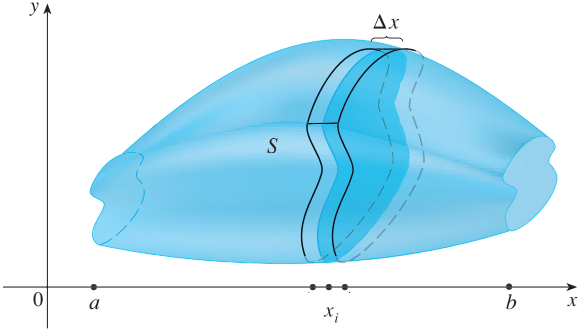

Since the area of the cross-section is $A(x)$, the volume of that (very short) cylinder is

$$\Delta V_i = A(x_i)\Delta x.$$

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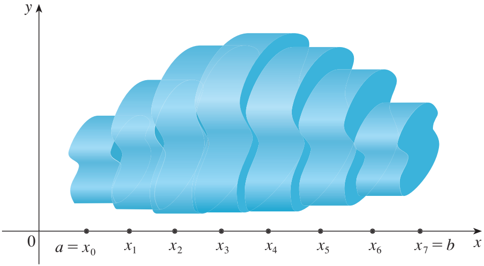
The volume of each cylinder is $\Delta V_i = A(x_i)\Delta x$. Summing up, we approximate the volume by

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x.$$



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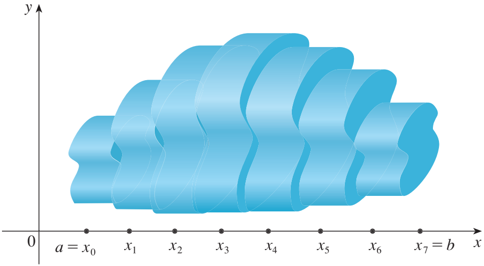

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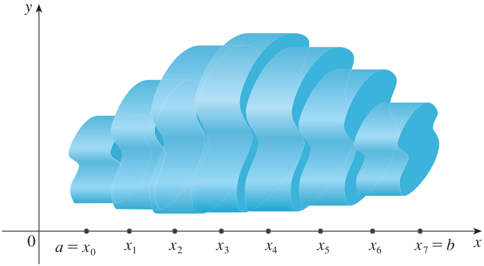

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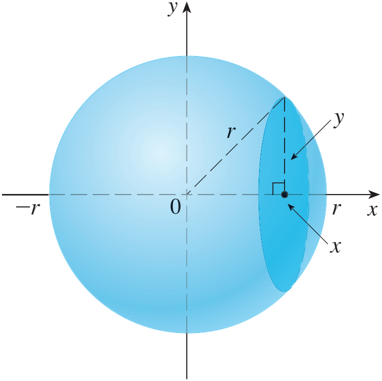
Place a sphere or radius r with its center at the origin:



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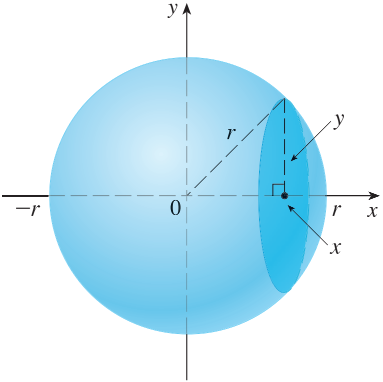

The cross-sections perpendicular to the x-axis are circles. For a fixed x, the cross-section's area depends on the radius $\rho(x)$. So what is the radius $\rho(x)$ of the corresponding circle??

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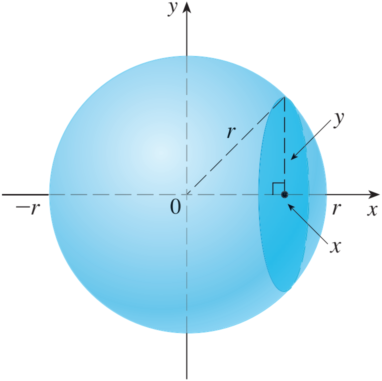

We have the triangle pictured, whose base is x (since that's how far out we are), whose hypotenuse is r (the radius of the whole sphere), and whose height is $y = \rho(x)$.

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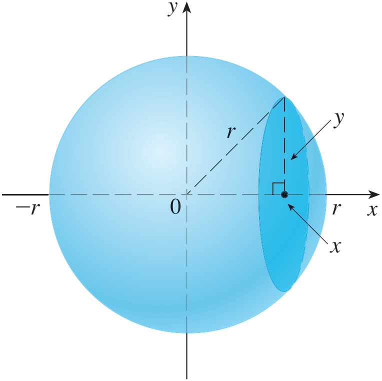

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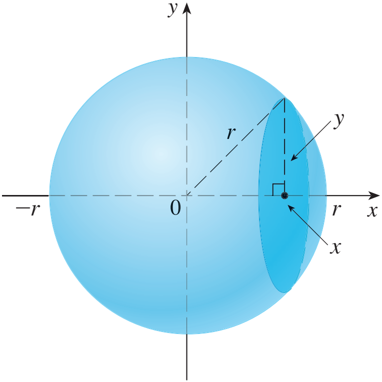

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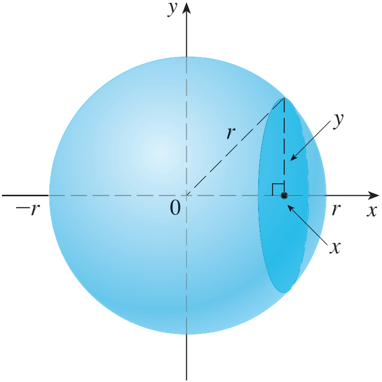

$A(x) = \pi(r^2 - x^2)$ (r is constant)

So using $V = \int_a^b A(x)dx$, we have

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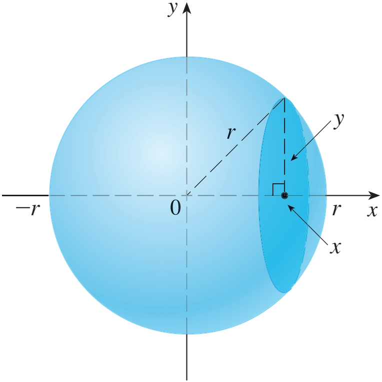

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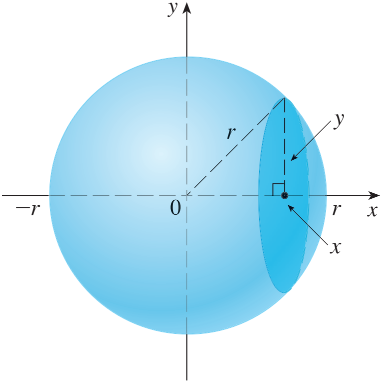

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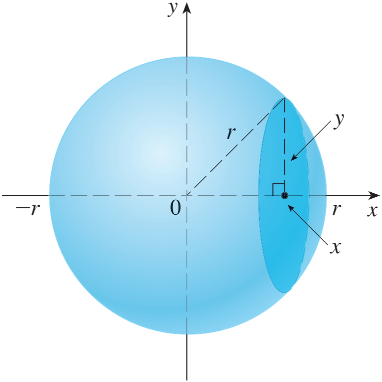

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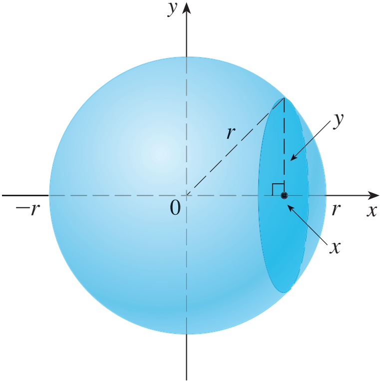

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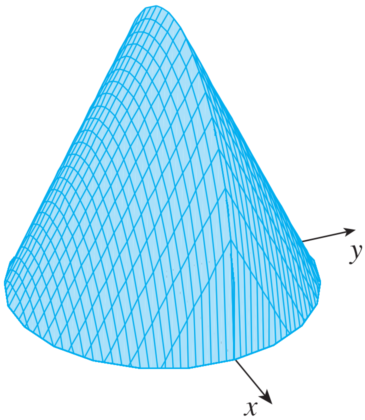

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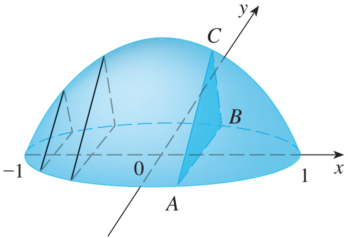

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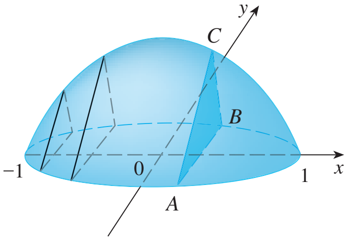

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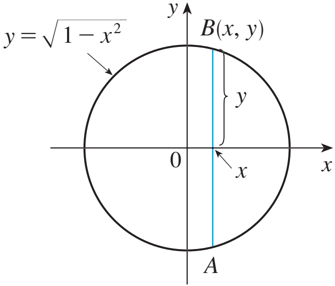

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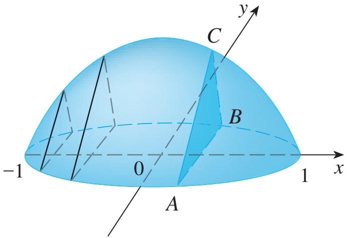

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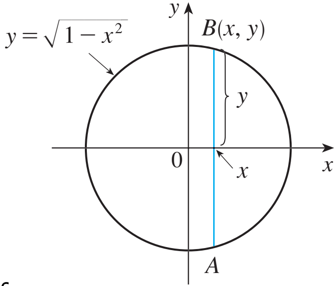
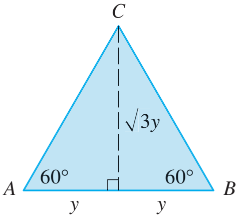


Slices: height Δx and triangular face



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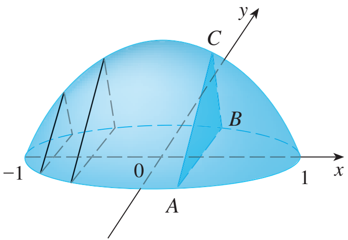

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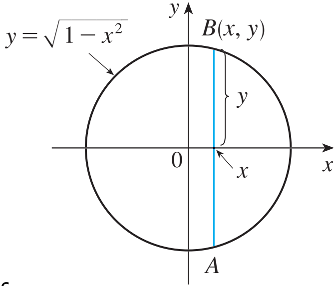

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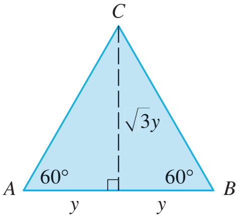

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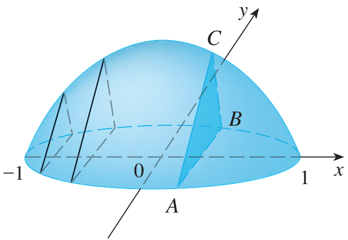

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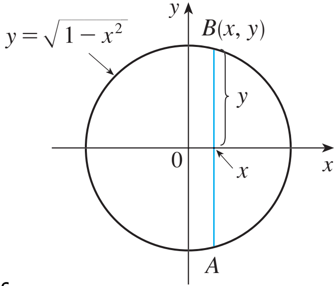

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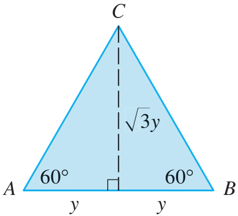

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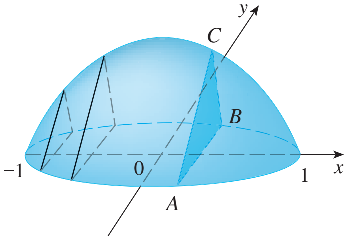

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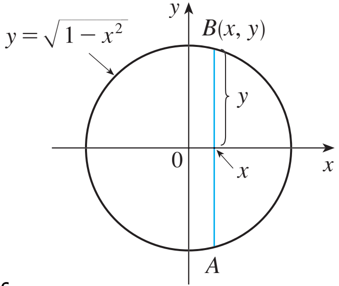

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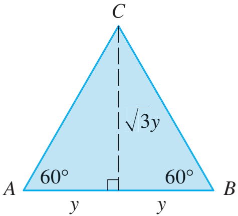

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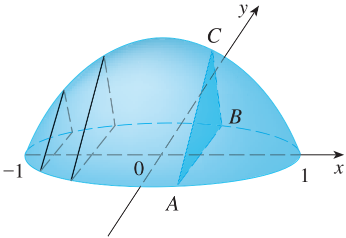

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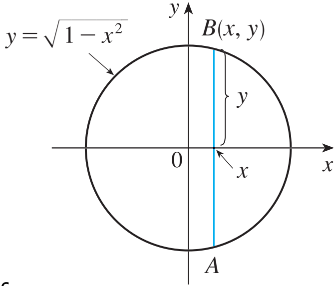
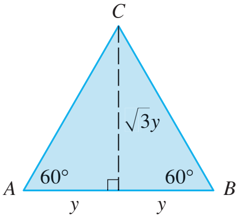


Slices: height Δx and triangular face



$$A(x) = \tfrac{1}{2}y * \sqrt{3}y = \frac{\sqrt{3}}{2}y^2$$
$$= \frac{\sqrt{3}}{2}(1 - x^2)$$

You try: What is V?

Volumes of revolution

Now say we have a volume that can be described as a 2-dimensional shape rotated around an axis.

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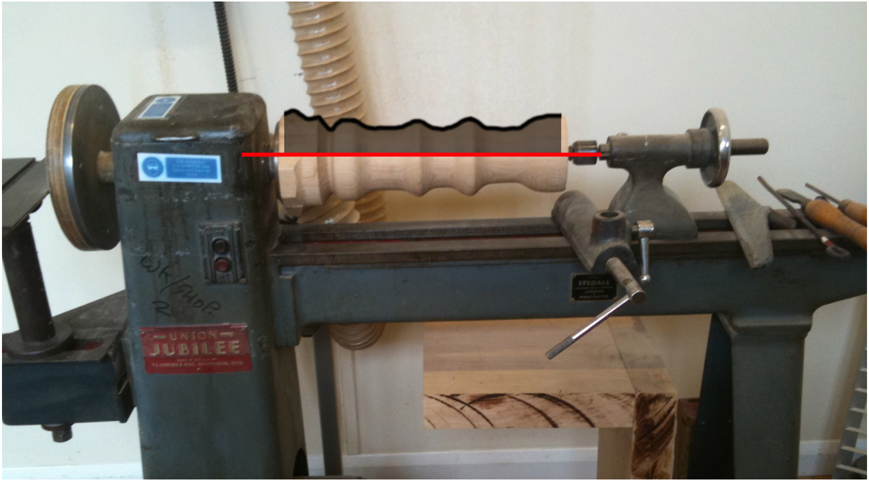

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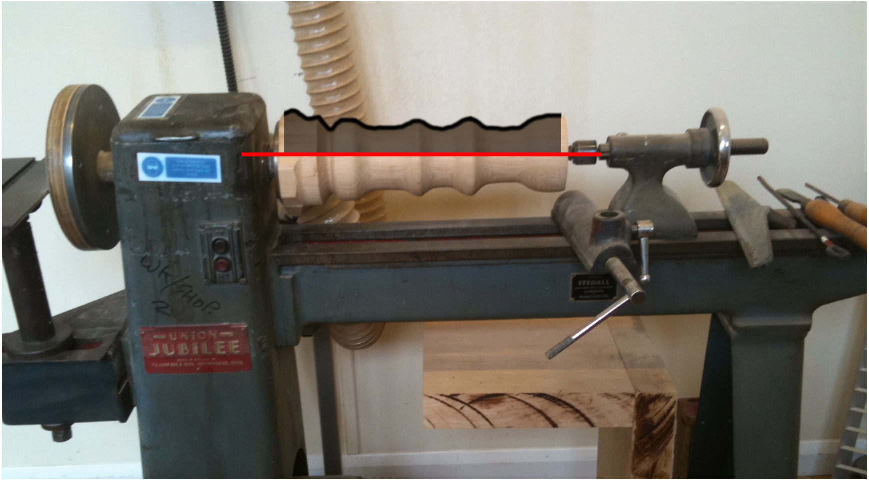

How much wood is left?

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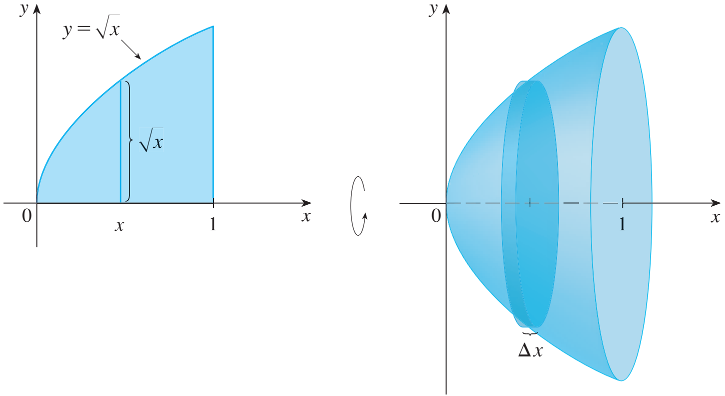

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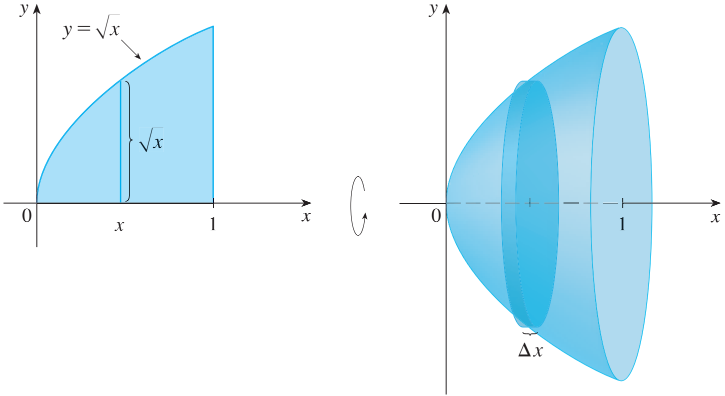

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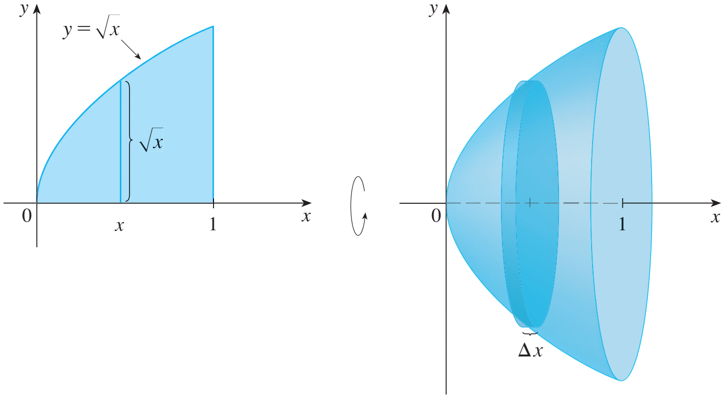

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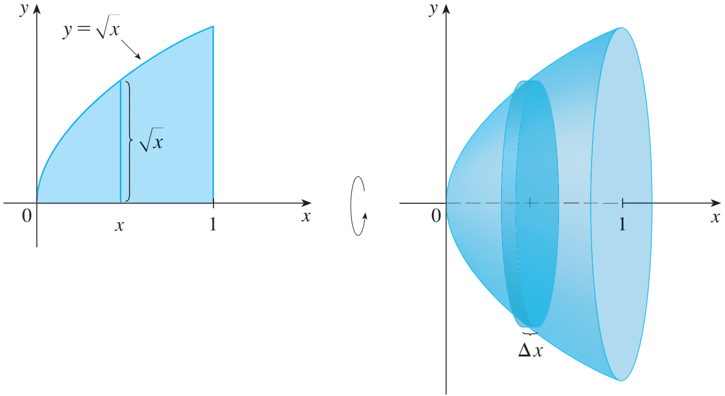

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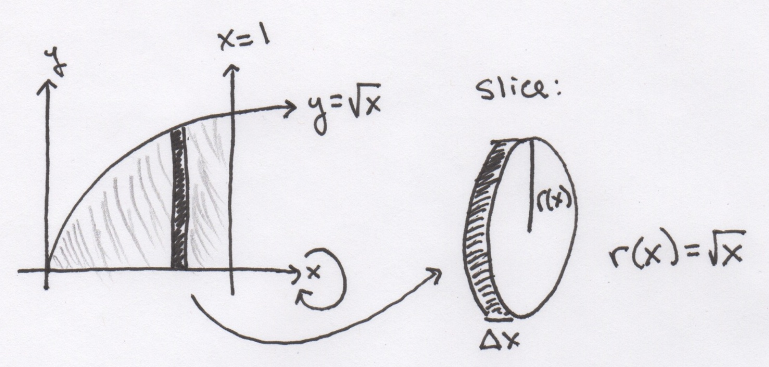
What is $A(x)$? What is V?

Answer:



$$A(x) = \pi r^2(x) = \pi(\sqrt{x})^2 = \pi x$$

So

$$V = \int_0^1 A(x)\ dx = \int_{0^1} \pi x\ dx = \tfrac{\pi}{3} x^3 \Big|_{x=0}^{1} = \tfrac{\pi}{3}.$$

Example

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

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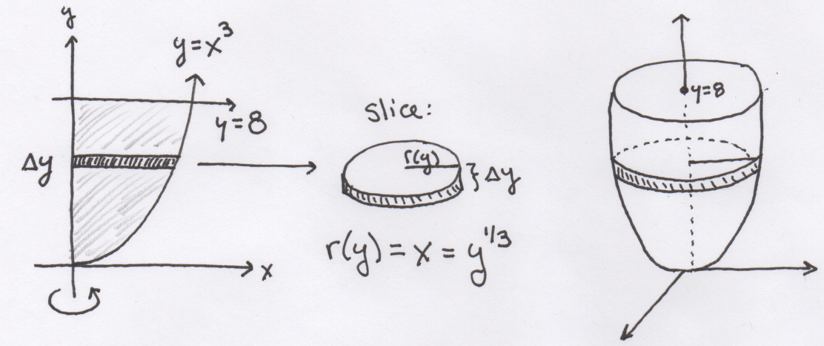

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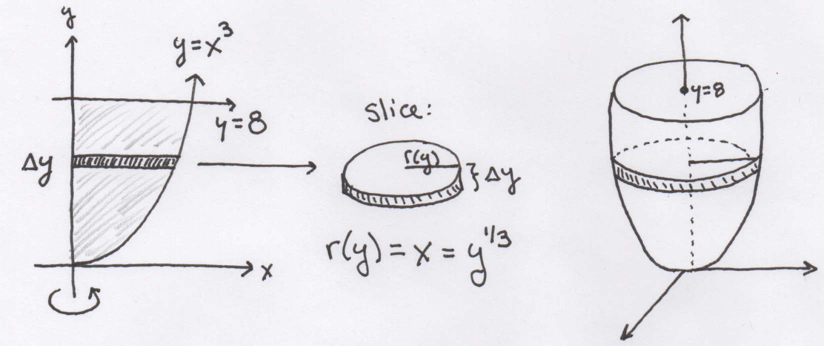

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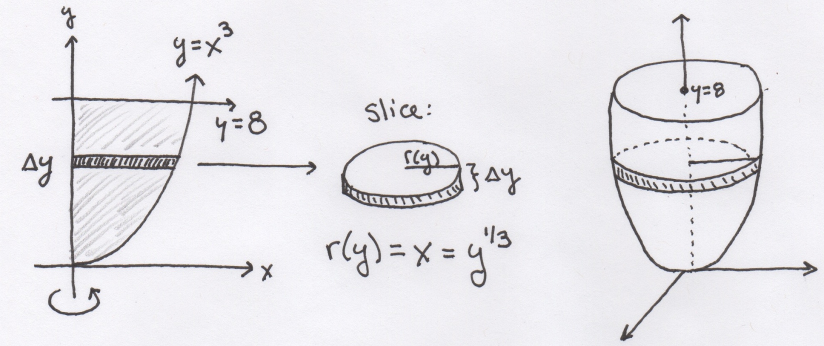

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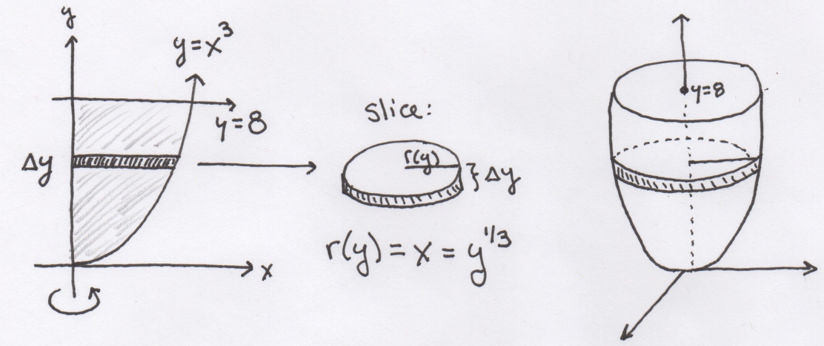

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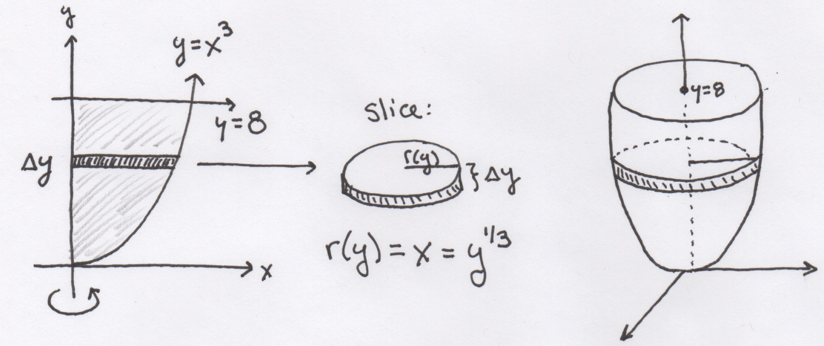

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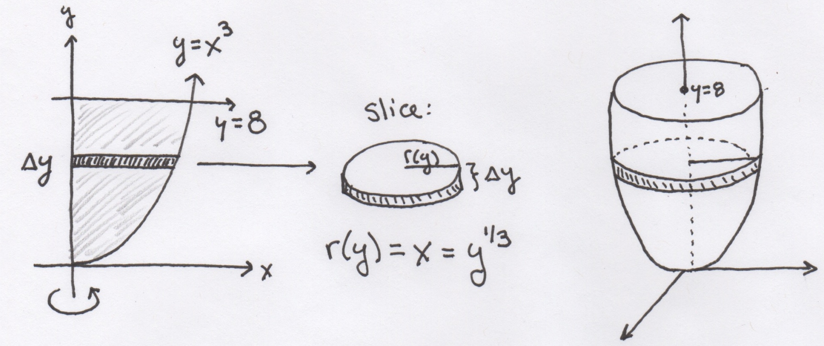

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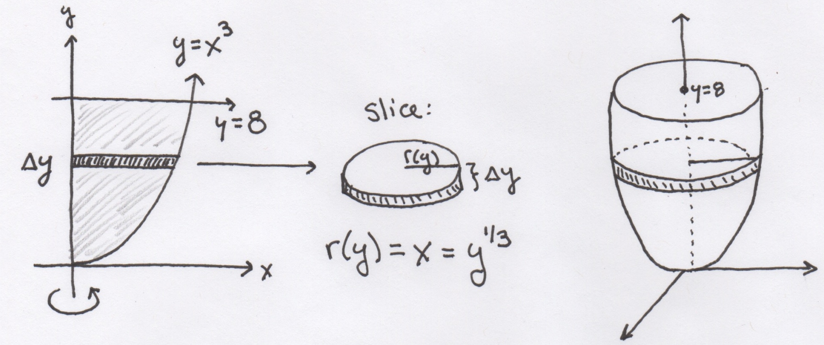

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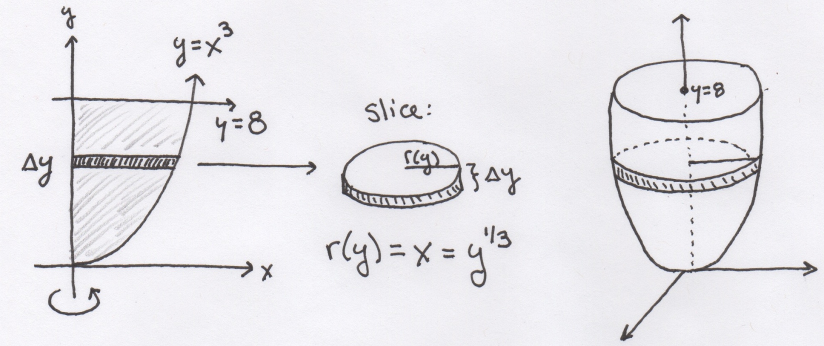

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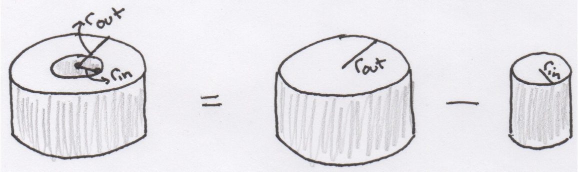
Washers: Volumes that are not convex

Notice that the volume of a hollowed out circular cylinder is

$$\text{Vol(Outer cylinder)} - \text{Vol(Inner cylinder)}$$



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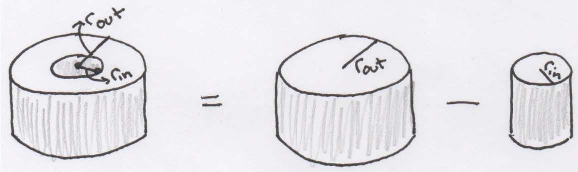

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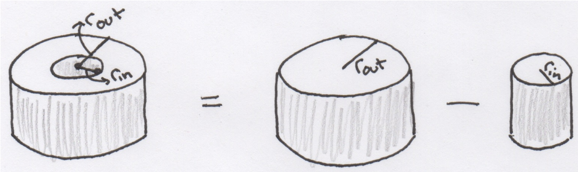

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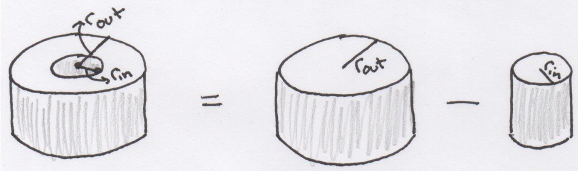

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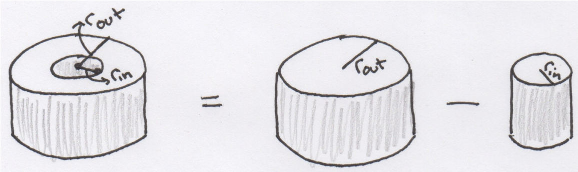

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$$A = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2,$$

so that $V = hA$ give the same answer as above!

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$$\boxed{V(\text{washer}) = \pi(r_{\text{out}}^2 - r_{\text{in}}^2)h.}$$

Example: Take the region bounded by $y = x$ and $y = x^2$, and rotate is around the x-axis. What is the resulting volume?

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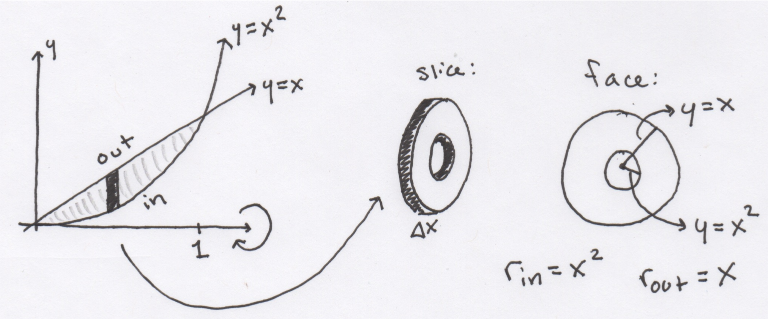

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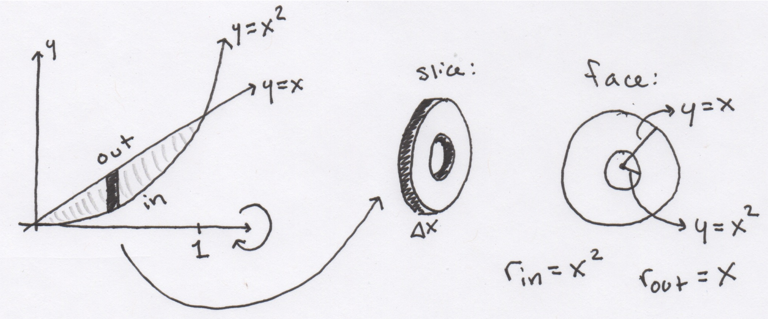

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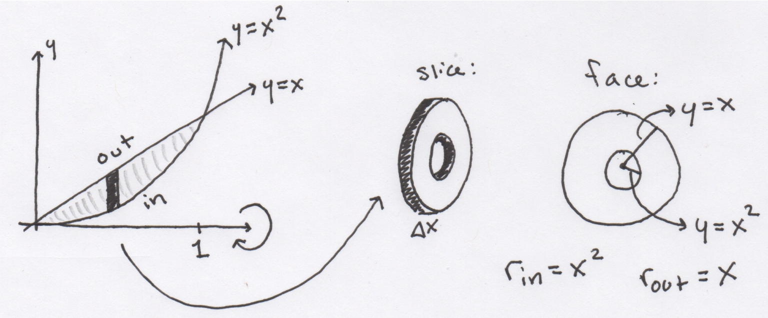

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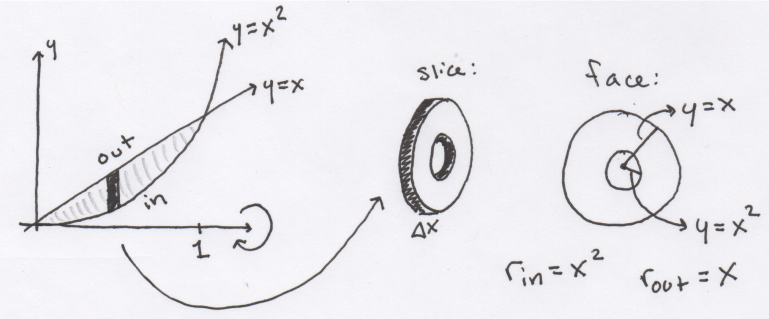

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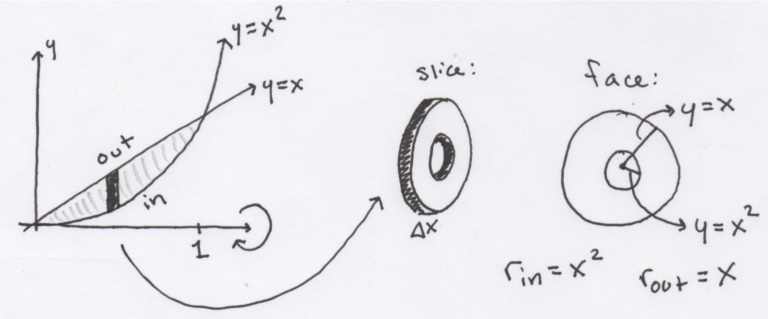

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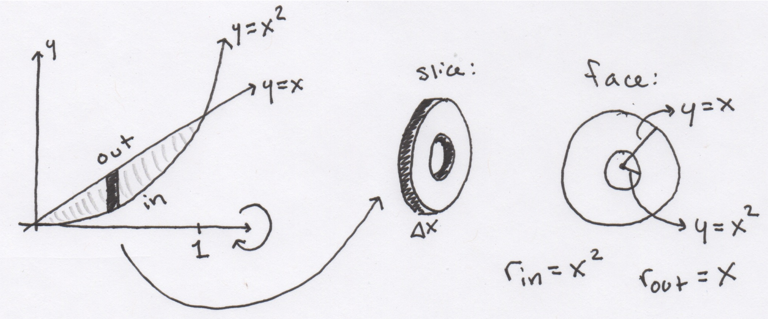

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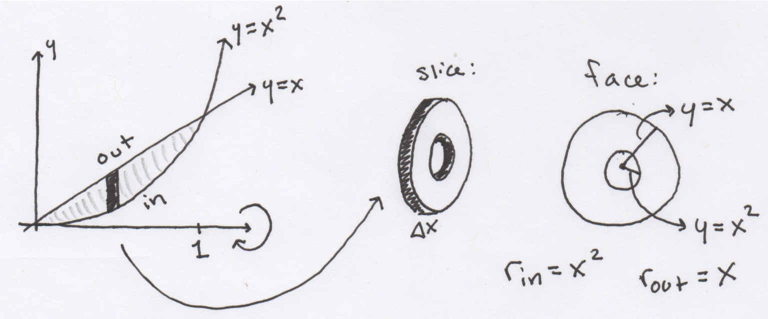

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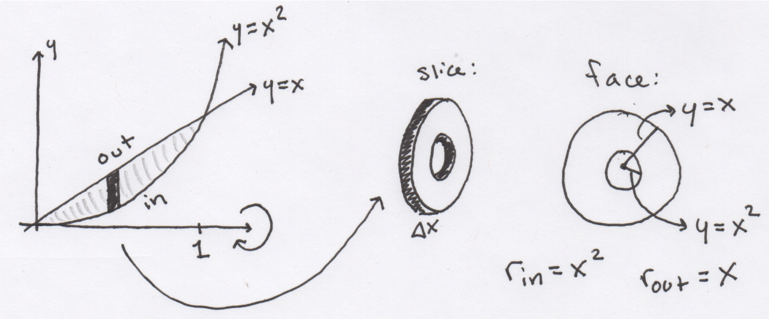

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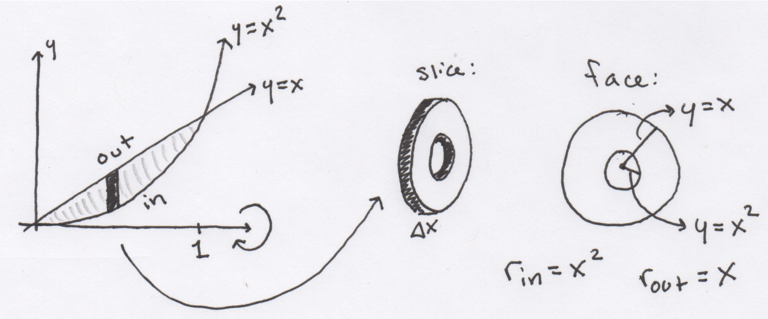

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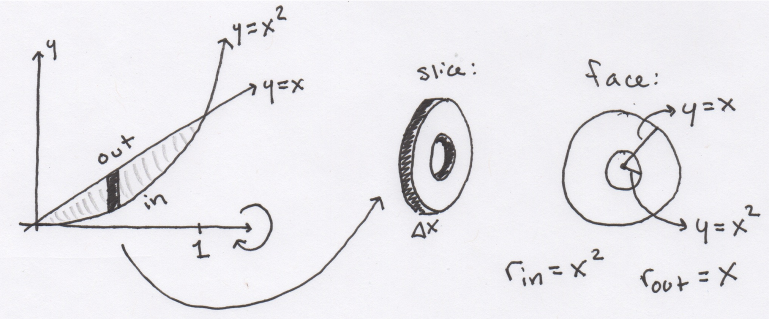

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You try

Take the region bounded by $y = 4$, $y = 4x^2$, and $x = 0$.

1. Draw the region $y = 4$, $y = 4x^2$, and $x = 0$. What are the end-points of this region in terms of x? in terms of y?
2. Rotating around the x-axis:
   (a) When you rotate this shape around the x-axis, what is the shape of the slices?
   (b) For each slice, what the height? What is the variable? Do I want $A(x)$ or $A(y)$?
   (c) What is $A(x)$ or $A(y)$ (whichever you chose above)?
   (d) What is the volume?
3. Rotating around the y-axis:
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General strategy for calculating volumes

1. What are your slices? So far, this is some sort of cylinder with face of area A and very small height.

2. What is your variable? This should be the thickness of your slice (Δx, Δy, Δz).
3. What are the endpoints with respect to the variable?
4. What is the area of the face of the slice (in terms of the variable)?

Integrate the area in part 4 versus the variable in part 2, between the endpoints in part 3.

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