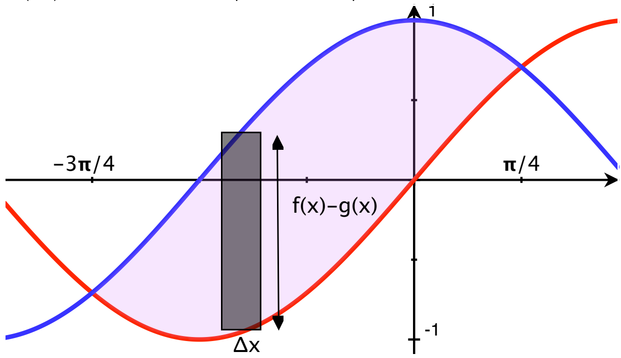
Today: 7.1 Area between curves

Suppose you want to calculate the area between $f(x) = \cos(x)$ and $g(x) = \sin(x)$ from $-3\pi/4$ to $\pi/4$:



Thinking back to Riemann sums, we can approximate the area using rectangles: Divide $[-3\pi/4, \pi/4]$ into n intervals. Let $\Delta x = (\pi/4 - (-3\pi/4))/n = \pi/n$, and let $x_i = -3\pi/4 + i\Delta x$. Then use rectangles with base Δx and height $f(x) - g(x)$ (since $f(x) = \cos(x)$ is on top and $g(x) = \sin(x)$ is on bottom).

$$\text{Area} \approx \sum_{i=1}^{n} \left(\frac{\pi}{n}\right) * (\cos(x_i) - \sin(x_i)).$$

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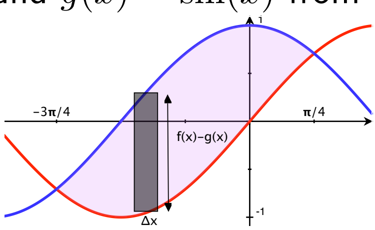


$\text{Area} \approx \sum_{i=1}^{n} \left(\frac{\pi}{n}\right) * (\cos(x_i) - \sin(x_i)).$

The larger the n, the better the approximation. So

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{\pi}{n}\right) * (\cos(x_i) - \sin(x_i)).$$

But the left-hand side is exactly the definition of $\int_{-3\pi/4}^{\pi/4} \cos(x) - \sin(x)\ dx$! So

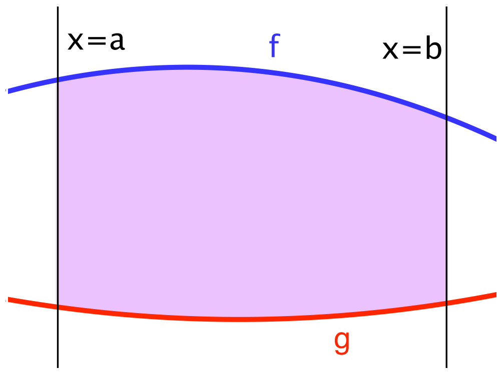
$$\text{Area} = \int_{-3\pi/4}^{\pi/4} \cos(x) - \sin(x)\ dx = \sin(x) + \cos(x)\Big|_{-3\pi/4}^{\pi/4}$$

$$= 2(\sqrt{2}/2) - 2(-\sqrt{2}/2) = 2\sqrt{2}.$$

Areas between curves, in general:

The area A bounded between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where $f(x) \geq g(x)$ over the interval $[a, b]$, is given by

$$A = \int_a^b f(x) - g(x)\, dx.$$

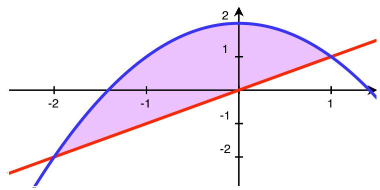


You try:

1. Graph $f(x) = -x^2 + 2$ and $g(x) = x$ on the same axes.
2. Verify that $f(x) \geq g(x)$ over the interval $[0, 1]$ and draw a new picture that shows the area bounded between $y = f(x)$, $y = g(x)$, $x = 0$ and $x = 1/2$. Compute this area.

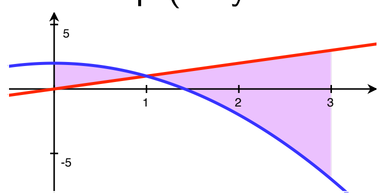
You try:

1. Notice that the graphs of $y = -x^2 + 2$ and $y = x$ intersect twice, at $x = -2$ and $x = 1$, and that there's an area that's trapped between these two functions:

   Set up the integral for the described area.

2. The area bounded by $y = -x^2 + 2$, $y = x$, $x = 0$ and $x = 3$ is actually in two pieces, one where $y = -x^2 + 2$ is on top and one where $y = x$ is on top (they switch at $x = 1$):

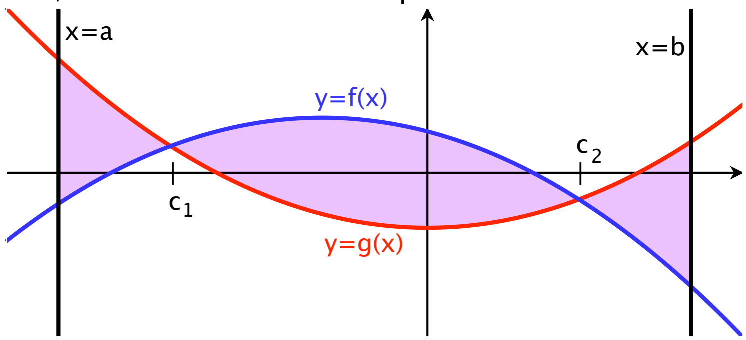
   

   What are the two integrals whose sum is the described area?

Areas between curves, in general:

If bounds $[a, b]$ are given:

1. If $f > g$ over the whole interval, $A = \int_a^b f(x) - g(x) dx$.
2. If the functions cross each other, compute where this happens, and break the area up:



$$A = \int_a^{c_1} g(x) - f(x)\, dx + \int_{c_1}^{c_2} f(x) - g(x)\, dx + \int_{c_2}^{b} g(x) - f(x)\, dx.$$

Areas between curves, in general:

If bounds are not given:

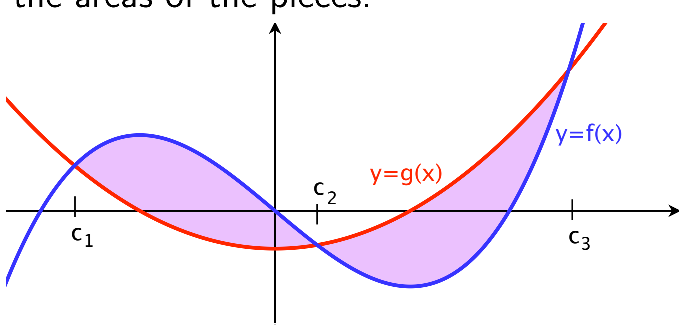
The problem will ask "find the area enclosed by the curves $y = f(x)$ and $y = g(x)$". This means that the two curves will cross enough times to define an enclosed area:

(a) Find the points of intersection $c_1, c_2, \ldots$.

(b) Decide which function is on top for each interval.

(c) If $f(x) > g(x)$ for over the interval $[c_i, c_{i+1}]$, the corresponding area is $A_i = \int_{c_i}^{c_{i+1}} f(x) - g(x)\, dx$.

(d) Add up the areas of the pieces.



$$A = \int_{c_1}^{c_2} f(x) - g(x)\, dx + \int_{c_2}^{c_3} g(x) - f(x)\, dx.$$

You try:

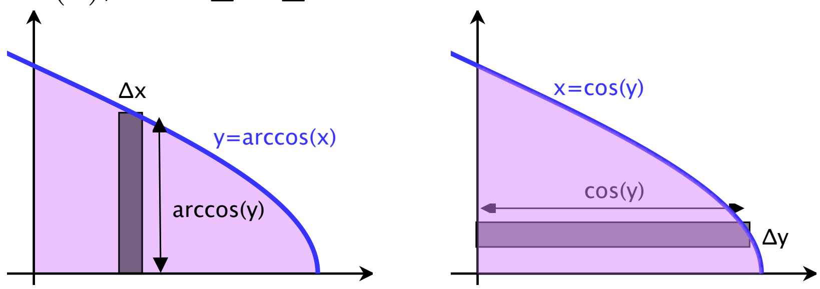
1. Graph $y = x^3$ and $y = x$ on the same axes.
2. Compute the area bounded by $y = x^3$, $y = x$, $x = 0$, and $x = 2$.
3. Compute the area enclosed by $y = x^3$ and $y = x$.

Sanity check: your answers should all be positive!
Always draw pictures!!!

Flipping the axes

Sometimes, because of the resulting integration problem, it can be better to calculate your integral versus y instead of versus x.

Example: Compute the area above the x-axis, below $y = \arccos(x)$, for $0 \le x \le 1$:



Instead of starting with rectangles that have base Δx and height $\arccos(x)$, resulting in the integral

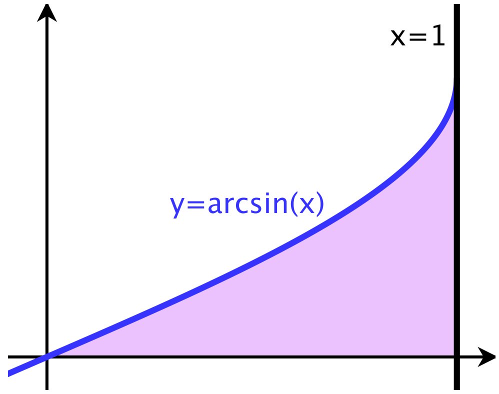
$$A = \int_0^1 \cos^{-1}(x)\ dx \quad \text{(use integration by parts)},$$

I could have rewritten $y = \arccos(x)$ as $x = \cos(y)$, and used rectangles that have height Δy and base $\cos(y)$, resulting in the integral

$$A = \int_0^{\pi/2} \cos(y)\ dy = \sin(y)\Big|_0^{\pi/2} = 1.$$

You try:

Calculate the area A bounded between $y = \arcsin(x)$, $y = 0$, and $x = 1$ in two ways:



1. Compute the standard way, with vertical rectangles, integrating versus x. You'll need to use integration by parts.
2. Compute using horizontal rectangles, integrating versus y. (Careful! In the previous example, the right function was $x = \cos(y)$ and the left function was $x = 0$. Now the right function is $x = 1$ and the left function is $x = \sin(y)$!)