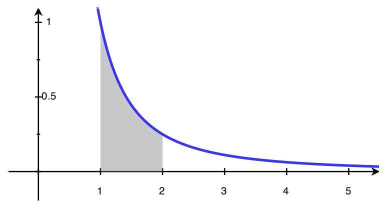
Today: 6.6 Improper integrals

Consider

$$\int_1^a x^{-2}\,dx, \quad \text{for } a = 2, 3, 4, 5.$$



$$\int_1^2 x^{-2}\,dx = -x^{-1}\Big|_{x=1}^{2} = -\frac{1}{2} + 1 = \frac{1}{2}$$

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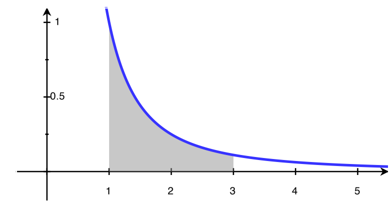

$$\int_1^3 x^{-2}\,dx = -x^{-1}\Big|_{x=1}^{3} = -\frac{1}{3} + 1 = \frac{2}{3}$$

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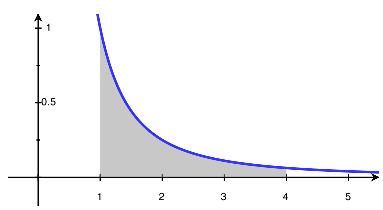

$$\int_1^4 x^{-2}\,dx = -x^{-1}\Big|_{x=1}^{4} = -\frac{1}{4} + 1 = \frac{3}{4}$$

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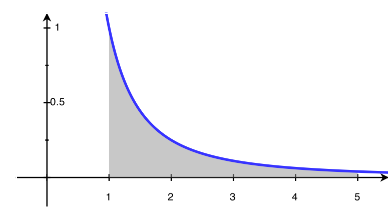

$$\int_1^5 x^{-2}\,dx = -x^{-1}\Big|_{x=1}^{5} = -\frac{1}{5} + 1 = \frac{4}{5}$$

Today: 6.6 Improper integrals

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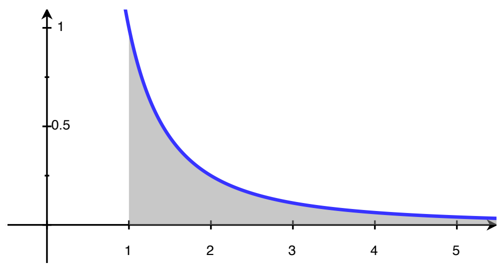


$$\int_1^n x^{-2}\,dx = -x^{-1}\big|_{x=1}^n = -\frac{1}{n} + 1 = \frac{n-1}{n}$$

Notice that

$$\lim_{n\to\infty}\int_1^n x^{-2}\,dx = \lim_{n\to\infty}\frac{n-1}{n} = 1.$$

So we say $\int_1^\infty x^{-2}\,dx = 1!$

Improper integrals: type 1 (infinite intervals)

1. If
   (a) $\int_a^t f(x)\,dx$ exists for every $t \geq a$, and
   (b) $\lim_{t\to\infty}\int_a^t f(x)\,dx$ exists,

   then
   $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$$
2. If
   (a) $\int_t^b f(x)\,dx$ exists for every $t \leq b$, and
   (b) $\lim_{t\to\infty}\int_t^b f(x)\,dx$ exists,

   then
   $$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$$

These integrals are called convergent if the limit exists, and divergent if the limit doesn't exist.

If $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are both convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

You try

1. Show that $\int_1^\infty \frac{1}{x}\, dx$ is divergent.
   [First calculate $\int_1^t \frac{1}{x}\, dx$. Then take the limit as $t \to \infty$.]

2. Calculate

   $$\int_0^\infty \frac{1}{1+x^2}\, dx, \quad \int_{-\infty}^0 \frac{1}{1+x^2}\, dx, \quad \text{and} \quad \int_{-\infty}^\infty \frac{1}{1+x^2}\, dx.$$

   (Recall, $\lim_{t\to\pm\infty} \tan^{-1}(t) = \pm\pi/2$.)
   [First calculate $\int_1^t \frac{1}{x}\, dx$ and $\int_t^1 \frac{1}{x}\, dx$.
   Then take the limits as $t \to \pm\infty$.]

You try

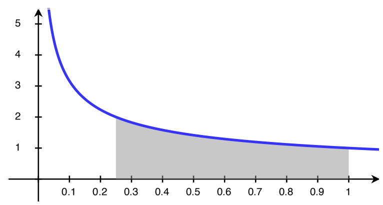
Decide which of the following converge (if so, to what) or diverge.

$$\int_0^\infty xe^x \, dx, \quad \int_{-\infty}^0 xe^x \, dx, \quad \text{and} \quad \int_{-\infty}^\infty xe^x \, dx.$$

Improper integrals: type 2 (discontinuous functions)

Example: Consider

$$\int_a^1 x^{-1/2}\,dx, \quad \text{for } a = 1/4, 1/9, 1/16, 1/25.$$



$$\int_{1/4}^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_{x=1/4}^1 = 2 - 2\cdot\tfrac{1}{2} = 1$$

Improper integrals: type 2 (discontinuous functions)

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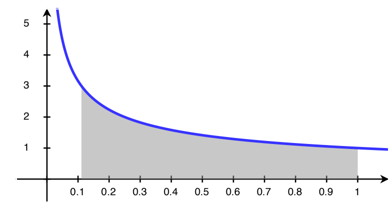

$$\int_{1/9}^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_{x=1/9}^1 = 2 - 2\cdot\tfrac{1}{3} = 2\cdot\frac{2}{3}$$

Improper integrals: type 2 (discontinuous functions)

Example: Consider

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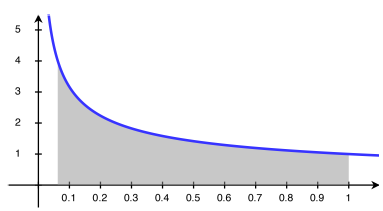

$$\int_{1/16}^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_{x=1/16}^1 = 2 - 2\cdot\tfrac{1}{4} = 2\cdot\frac{3}{4}$$

Improper integrals: type 2 (discontinuous functions)

Example: Consider

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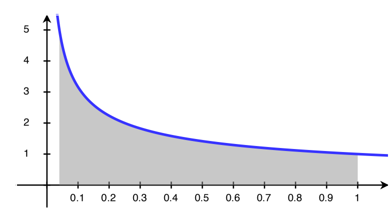

$$\int_{1/25}^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_{x=1/25}^1 = 2 - 2\cdot\tfrac{1}{5} = 2\cdot\frac{4}{5}$$

Improper integrals: type 2 (discontinuous functions)

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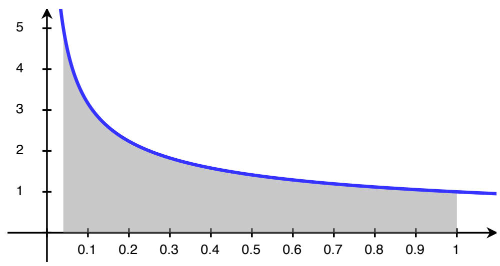


$$\int_{1/n^2}^1 x^{-1/2}\, dx = 2x^{1/2}\Big|_{x=1/n^2}^1 = 2-2\cdot\tfrac{1}{n} = 2\cdot\frac{n-1}{n} \to 2 \text{ as } n \to \infty$$

So we say

$$\int_0^1 x^{-1/2}\, dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\, dx = \lim_{t\to 0^+} 2 - 2\cdot\sqrt{t} = 2.$$

Improper integrals: type 2 (discontinuous functions)

1. If
   (a) $f(x)$ is continuous on $[a, b)$ but is discontinuous at b, and
   (b) $\lim_{t\to b^-} \int_a^t f(x)\, dx$ exists,

   then
   $$\int_a^b f(x)\, dx = \lim_{t\to b^-} \int_a^t f(x)\, dx.$$
2. If
   (a) $f(x)$ is continuous on $(a, b]$ but is discontinuous at a, and
   (b) $\lim_{t\to a^+} \int_t^b f(x)\, dx$ exists,

   then
   $$\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^b f(x)\, dx.$$

These integrals are called convergent if the limit exists, and divergent if the limit doesn't exist. If $f(x)$ has a discontinuity at c, where $a < c < b$, and $\int_b^c f(x)\, dx$ and $\int_c^a f(x)\, dx$ are both convergent, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

You try:

1. Show $\int_2^5 \frac{1}{x-2}\ dx$ converges to $2\sqrt{3}$.
   [First calculate $\int_t^5 \frac{1}{x-2}\ dx$. Then take the limit as $t \to 2^+$.]
2. Show $\int_0^3 \frac{1}{x-1}\ dx$ diverges.
   [Break it up into $\int_0^1 \frac{1}{x-1}\ dx + \int_1^3 \frac{1}{x-1}\ dx$.]
3. Decide whether $\int_0^{\pi/2} \sec(x)\ dx$ diverges or converges.
   [Recall $\sec(x)$ has vertical asymptotes at $\pi/2 + k\pi$.]

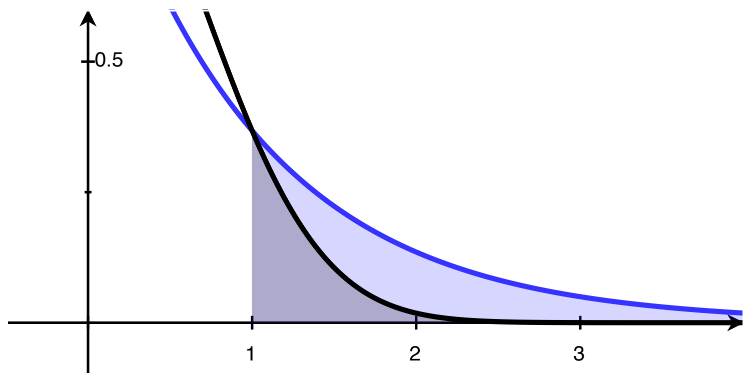
Comparison test for improper integrals

Example: Show that $\int_1^\infty e^{-x^2}\ dx$ is convergent.
Recall, we can't calculate $\int e^{-x^2}\ dx$ exactly.
However, for $x > 1$ we have

$$x^2 > x, \quad \text{so that } 1/e^{x^2} > 1/e^x.$$

Also, since $e^{-x^2} > 0$, we have $0 < \int_1^\infty e^{-x^2}\ dx$. Therefore

$$0 < \int_1^\infty e^{-x^2}\ dx < \int_1^\infty e^{-x}\ dx.$$



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But

$$\int_1^\infty e^{-x}\ dx = \lim_{t\to\infty} \int_1^t e^{-x}\ dx$$

$$= \lim_{t\to\infty} -e^{-x}\Big|_1^t = \lim_{t\to\infty} -e^{-t} + 1 = 0 + 1 = 1.$$

So

$$0 < \int_1^\infty e^{-x^2}\ dx < 1,$$

and is therefore convergent. (Still can't calculate exactly)

You try

Decide whether

$$\int_0^{\pi} \frac{\cos^2(x)}{\sqrt{x}}\ dx$$

converges or diverges. [Hint: $-1 \leq \cos(x) \leq 1$]