Warm up: Recall we can approximate $\int_a^b f(x)\,dx$ using rectangles as follows:

i. Pick a number n and divide $[a, b]$ into n equal intervals. Note that $\Delta x = (b-a)/n$ is the length of each of these intervals.

ii. Choose a point c in each of the intervals (usually either the left-most point, the right-most point, or the mid point).

iii. Use a rectangle with base $(b-a)/n$ and height $f(c)$ to model the area under the curve $y = f(x)$ over each of the intervals.

iv. Add up the area of the rectangles.

Now consider $I = \int_1^4 x^2\,dx$.

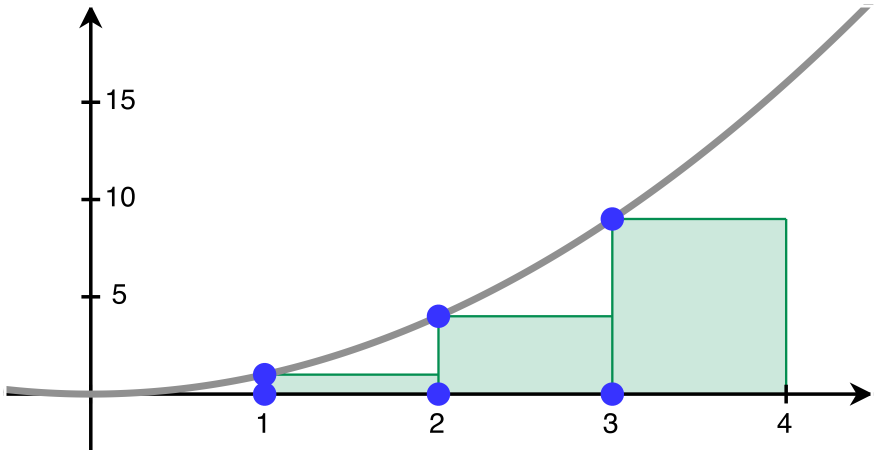
Approximate I using the given n and c, and draw a picture to go with that shows (a) $y = x^2$, (b) the n intervals on the x-axis, (c) the point c in each of the intervals, and (d) the rectangle that approximates the area under the curve.

(1) $n = 3$ and c being the left-most point of each of the intervals.

(2) $n = 6$ and c being the right-most point of each of the intervals.

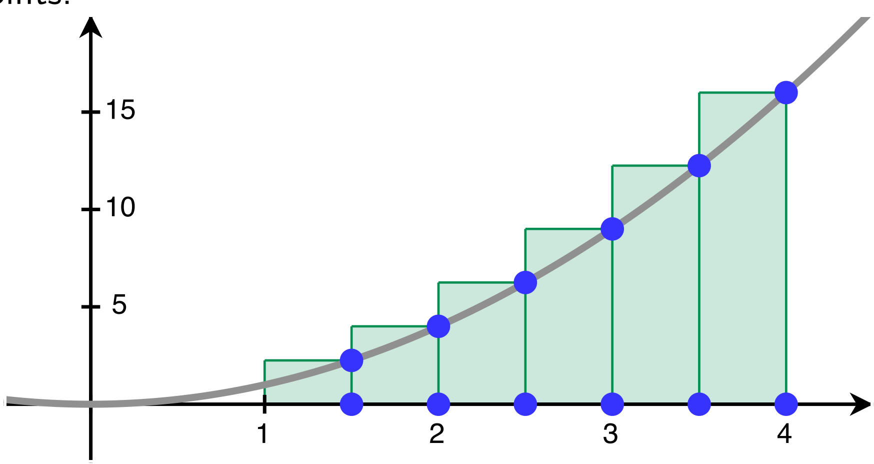
(3) $n = 2$ and c being the midpoint of each of the intervals.

Approximating $I = \int_1^4 x^2 \; dx$ with $n = 3$ intervals using left endpoints:



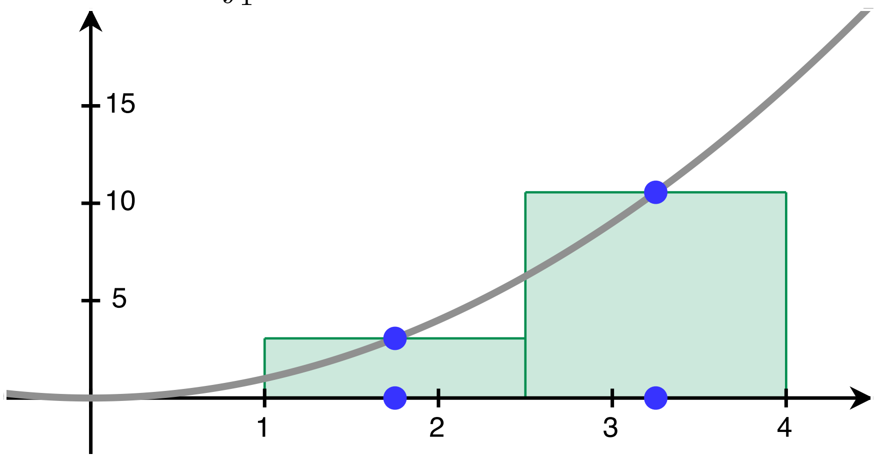
$$I \approx 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 = 14$$

Approximating $I = \int_1^4 x^2 \; dx$ with $n = 6$ intervals using right endpoints:



$$I \approx .5 \cdot 1.5^2 + .5 \cdot 2^2 + .5 \cdot 2.5^2 + .5 \cdot 3^2 + .5 \cdot 3.5^2 + .5 \cdot 4^2 = 24.875$$

Approximating $I = \int_1^4 x^2\ dx$ with $n = 2$ intervals using midpoints:



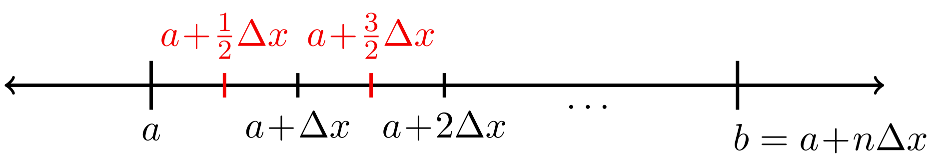
$$I \approx 1.5 \cdot 1.75^2 + 1.5 \cdot 3.25^2 = 20.4375$$

Review from Section 4.2

Approximating $I = \int_a^b f(x)\ dx$ using n intervals:
The intervals are of length $\Delta x = (b-a)/n$ and

$$I \approx \sum_{i=1}^{n} \Delta x * f(c_i),$$

where c_i is...

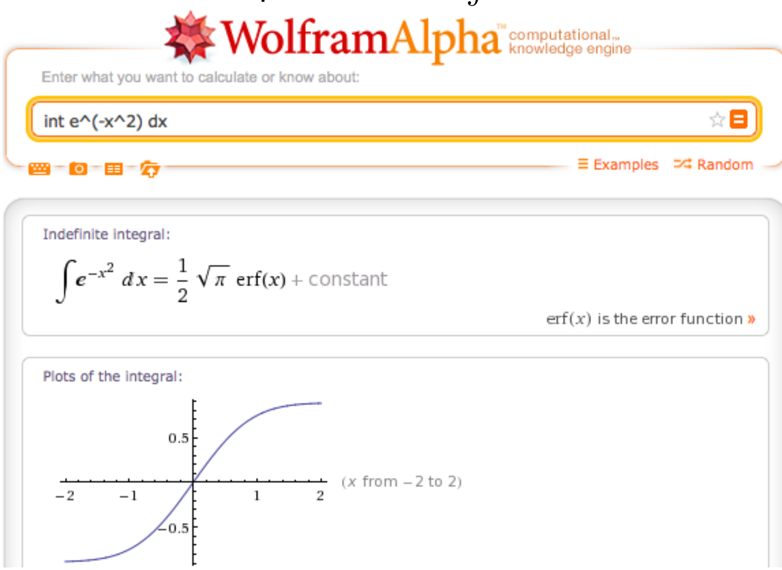


Left-hand endpoints: $c_i = a + (i-1)\Delta x$
Right-hand endpoints: $c_i = a + i\Delta x$
Midpoints: $c_i = a + (i - \frac{1}{2})\Delta x$

Section 6.5: Approximate integration

Why would we need approximations now that we have a bunch of fancy new integration techniques?

Example: What is $\int e^{-x^2}\,dx$?

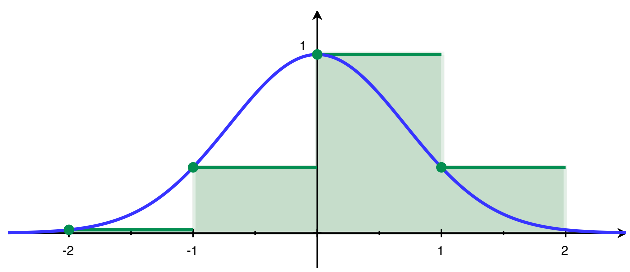


From Wikipedia: "In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations."

Approximating $\int_{-2}^{2} e^{-x^2}\,dx$ using rectangles

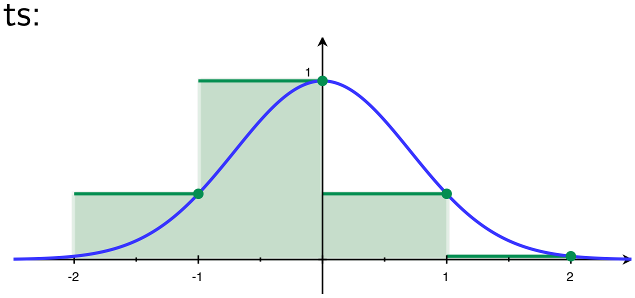
Let $n = 4$ (so that $\Delta x = (2-(-2))/4 = 1$).

Left endpoints:



$$I \approx e^{-4} + e^{-1} + e^{0} + e^{-1} = 1.7540\ldots$$

Right endpoints:

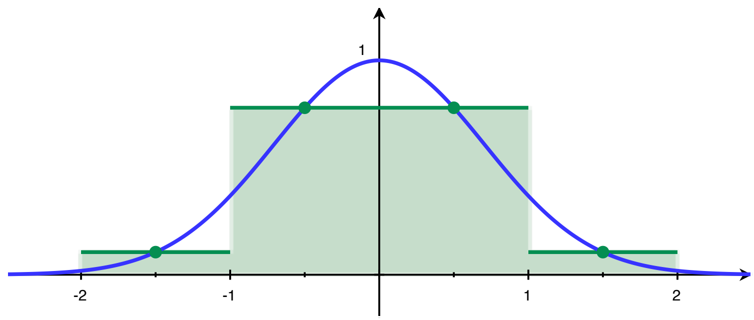


$$I \approx e^{-1} + e^{0} + e^{-1} + e^{-4} = 1.7540\ldots$$

Approximating $\int_{-2}^{2} e^{-x^2} dx$ using rectangles

Let $n = 4$ (so that $\Delta x = (2 - (-2))/4 = 1$).

Midpoints:



$$I \approx e^{-(-1.5)^2} + e^{-(-.5)^2} + e^{-(.5)^2} + e^{-(1.5)^2} = 1.7684\ldots$$

Error

Let L_n, R_n, and M_n be the estimates of a definite integral with n intervals, using left, right, and midpoints, respectively.
For example, for the definite integral $\int_{-2}^{2} e^{-x^2}\ dx$,

$$L_4 = 1.7540\ldots, \quad R_4 = 1.7540\ldots, \quad \text{and} \quad M_4 = 1.7684\ldots.$$

In reality,

$$\int_{-2}^{2} e^{-x^2}\ dx = 1.7641\ldots.$$

So the errors for L_4 and R_4 were about 0.01, and the error for M_4 was about -0.004.
The reason I can tell you, definitively, to 4 digits, the value of this integral is because we also have formulas for upper bounds on the error using various approximation methods. This bound depends on things like the number of intervals (the more the better), the length of the total interval we integrate over (the smaller the better), and the curvature (second derivative) of the function (the flatter the better).

Error for the midpoint rule

Suppose we approximate $\int_a^b f(x)\,dx$ using n intervals and midpoints. The error of the approximation M_n is exactly

$$E_M = \int_a^b f(x)\,dx - M_n.$$

Calculate $f''(x)$. Find a smallest value $0 \leq K$ where you can calculate that $|f''(x)| \leq K$ over the interval $[a, b]$.
Example: Calculating $\int_1^4 x^2\,dx$. We calculated $M_2 = 20.4375$.
Now, $f''(x) = 2$. So let $K = 2$.
Then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Example continued: $K = 2$, $b - a = 3$ and $n = 2$. So

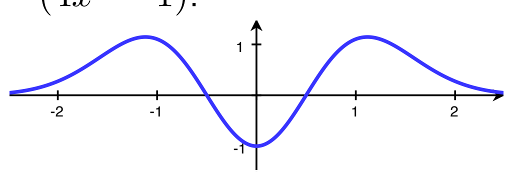
$$|E_M| \leq \frac{2(3)^3}{24 \cdot 2^2} = 9/16 = 0.5625.$$

Comparing against the exact value, $\int_1^4 x^2\,dx = 21$. So $E_M = 21 - 20.4375 = 0.5625$. So our bound was exact!

Error for the midpoint rule

$$\boxed{|E_M| \leq \frac{K(b-a)^3}{24n^2} \text{ where } |f''(x)| \leq K \text{ over } [a, b].}$$

Another example: Calculating $\int_{-2}^2 e^{-x^2}$.We showed $M_4 \approx 1.7684$.
Here, $f''(x) = e^{-x^2}(4x^2 - 1)$:



Max value: $4/e^{5/4} = 1.1460\ldots$, Min value: -1.
So let $K = 1.1461$.

Then since $b - a = 4$ and $n = 4$,
$|E_M| \leq \frac{(1.1461)4^3}{24 \cdot 4^2} = 1.1910 \cdots \leq 1.1911$.
Checking against exact values:

$$E_M \approx 0.004 \leq 1.1911 \checkmark.$$

You try:

$$M_n = \sum_{i=1}^{n} \Delta x * f(c_i), \text{ where } \Delta x = (b-a)/n \text{ and } c_i = a + (i - \tfrac{1}{2})\Delta x$$

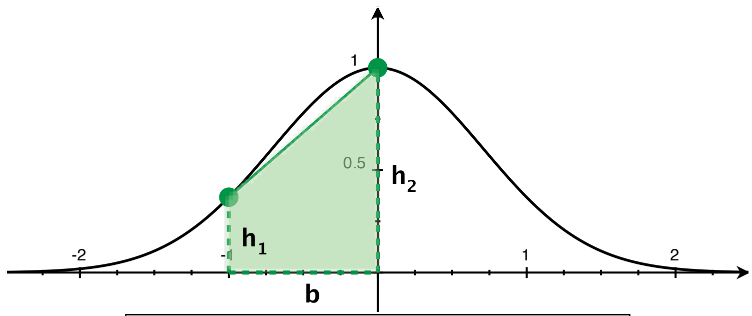
$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \text{ where } |f''(x)| \leq K \text{ over } [a, b].$$

1. Use the midpoint rule to approximate $\int_{-1}^{2} x^4 \, dx$ using $n = 3$. Draw a picture to help yourself.
2. Calculate $\frac{d^2}{dx^2}x^4$ and maximize $|\frac{d^2}{dx^2}x^4|$ over $[-1, 2]$. Let K be that maximum value.
3. Calculate an upper bound on E_M using the formula above.
4. Calculate $\int_{-1}^{2} x^4 \, dx$ exactly, and use that to calculate E_M exactly. Compare to your bound.

Approximations using other shapes: Trapezoids!

Instead of picking one height over each interval (approximating the function as a constant) we can pick a sloped line over each interval (approximating the function as a line) and use a trapezoid to approximate the area under the curve. Use the line that intersects the function at both endpoints of each interval.

Example: Approximate $\int_{-2}^{2} e^{-x^2}\, dx$ using trapezoids with $n = 4$.



Area (trapezoid) $= b * \frac{h_1+h_2}{2}$

For example: $b = 1$, $h_1 = f(-1)$, $h_2 = f(0)$,

so $A_2 = 1 * \frac{f(-1)+f(0)}{2}$

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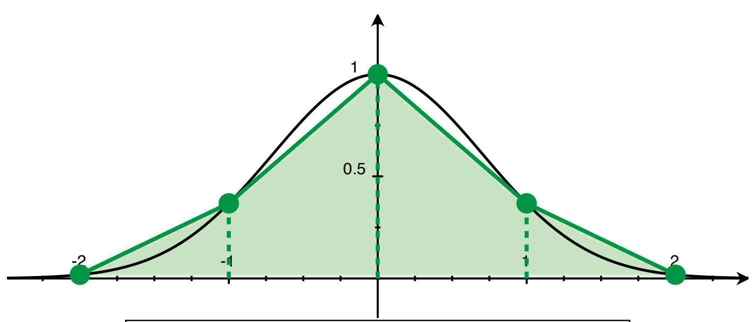

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For example: $b = 1$, $h_1 = f(-1)$, $h_2 = f(0)$,

so $A_2 = 1 * \frac{f(-1)+f(0)}{2}$

$$A = 1 * \frac{f(-2)+f(-1)}{2} + 1 * \frac{f(-1)+f(0)}{2} + 1 * \frac{f(0)+f(1)}{2} + 1 * \frac{f(1)+f(2)}{2}$$

$$= \tfrac{1}{2} * \left[f(-2) + f(2) + 2\big(f(-1) + f(0) + f(1)\big)\right]$$

In general,

$$T_n = \tfrac{1}{2}\Delta x(f(c_0) + f(c_n) + 2(f(c_1) + f(c_2) + \cdots + f(c_{n-1})))$$

where $\Delta x = (b-a)/n$ and $c_i = a + i\Delta x$.

You try:

1. Draw a graph of $f(x) = x^4$ over $[-1, 2]$.
2. Let $n = 3$ and calculate Δx and $c_i = a + i\Delta x$ for $i = 0, 1, 2,$ and 3. Mark the c_i's on the x-axis.
3. Mark the 4 points on the graph corresponding to $f(c_i)$.
4. Draw the three trapezoids whose tops are the line segments joining $f(c_{i-1})$ to $f(c_i)$.
5. Calculate the areas of the three trapezoids.
6. Add the areas together to get T_n.
7. Use the formula $T_n = \frac{1}{2}\Delta x(f(c_0) + f(c_n) + 2(f(c_1) + f(c_2) + \cdots + f(c_{n-1})))$ and compare to your previous answer (you should get the same thing).
8. Compare your answer to the exact value of $\int_{-1}^{2} x^4 \, dx$.

Trapezoid error

Let K be such that $|f''(x)| \leq K$ over $[a, b]$ as before. Then the error

$$E_T = \int_a^b f(x)\ dx - T_n$$

is bounded above by

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

(Recall $|E_M| \leq \frac{K(b-a)^3}{24n^2}$.)

You try: Give an upper bound for E_T for our estimate T_3 of $\int_{-1}^{2} x^4\ dx$.

Simpson's rule

Rectangles are like approximating $f(x)$ as a constant. (Needed one point over each interval.)
Trapezoids are like approximating $f(x)$ as a line. (Needed two points over each interval.)
Simpson's rule is approximating $f(x)$ as a parabola.

A generic parabola is given by $a_0 + a_1 x + a_2 x^2$. So we need three points to define a parabola.

So over each interval, take (1) the left endpoint, (2) the midpoint, and (3) the right endpoint, and find the parabola that passes through $f(x)$ above those three points. Actually, caution!! The book's convention is to call this $2n$ intervals, and pick one parabola for every two intervals.

Simpson's rule

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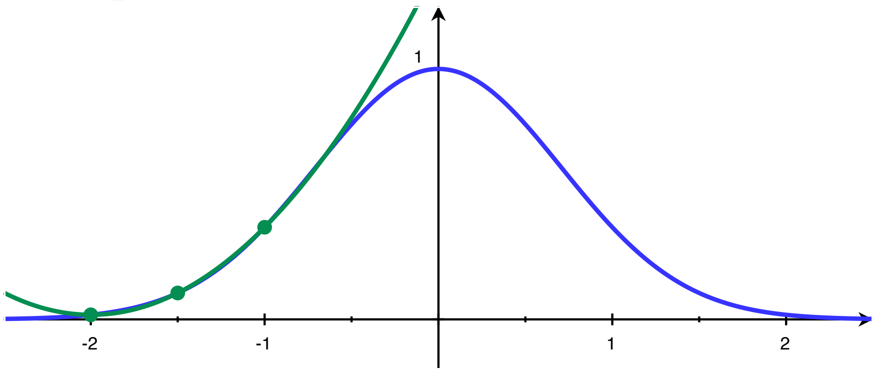
Example: $\int_{-2}^{2} e^{-x^2}\ dx$. Calculate S_8. (This has 4 parabolas for $n = 8$.)



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So over each n intervals, take (1) the left endpoint, (2) the midpoint, and (3) the right endpoint, and find the parabola that passes through $f(x)$ above those three points. Actually, caution!! The book's convention is to call this $2n$ intervals, and pick one parabola for every two intervals.

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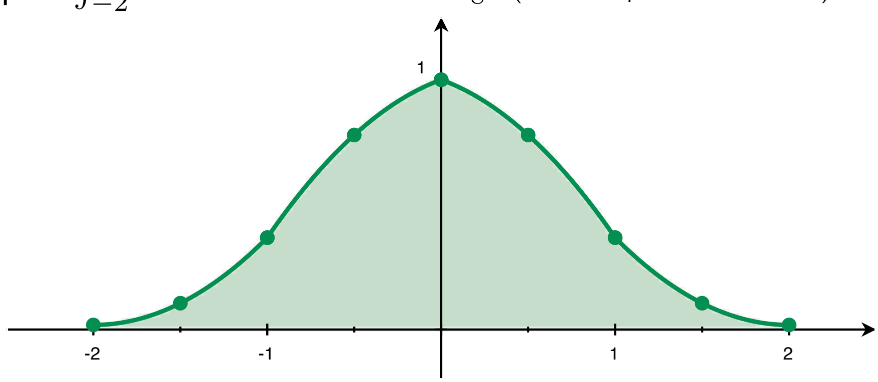

Whatever a_0, a_1, and a_2 are, we can calculate

$$\int_{c_{i-1}}^{c_i} a_0 + a_1 x + a_2 x^2\ dx = a_0 x + \tfrac{1}{2} a_1 x^2 + \tfrac{1}{3} a_2 x^3 \Big|_{c_{i-1}}^{c_i}$$

Simpson's rule

Let n be even. The resulting approximation, once the curves are fit and the integrals are taken, gives

$$S_n = \frac{1}{3}\Delta x\big(f(c_0) + 4f(c_1) + 2f(c_2) + 4f(c_3) + \cdots + 2f(c_{n-2}) + 4f(c_{n-1}) + f(c_n)\big)$$

where $\Delta x = (b-a)/n$ and $c_i = a + i\Delta x$. (Read pp 351–353 in the book)

You try: Approximate $\int_{-1}^{2} x^4 \, dx$ using S_6.

Error: For $E_S = \int_a^b f(x)\, dx - S_n$ and $K \geq |f^{(4)}(x)|$ over $[a, b]$ (new K!!),

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

You try: Calculate an upper bound for $|E_S|$ for $\int_{-1}^{2} x^4 \, dx$ and $n = 6$. Compare to the exact value of $|E_S|$.

Consider $\int_0^\pi \sin(x)$.

1. Calculate the maximum value of $\left|\frac{d^2}{dx^2}\sin(x)\right|$ over $[0, \pi]$. Let this be K.
2. For each of M_4, T_4 and S_4, do the following:
   (a) Draw a picture of the approximation, with $y = \sin(x)$ overlaid.
   (b) Calculate the approximation.
   (c) Calculate an upper bound of the error of the approximation.
   (d) Compare your upper bound against the actual value of the error.