Today: 6.3 Partial fractions

Warm up: Recall that if you require *real* coefficients (not complex), polynomials always factor into degree 1 or 2 parts. For example,

$$x^3 + x^2 + 3x - 5 = \underbrace{(x-1)(x+1-2i)(x+1+2i)}_{\text{allowing complex coeffs}} = \underbrace{(x-1)(x^2+2x+5)}_{\text{using only real coeffs}}$$

1. Factor the following polynomials (as far as possible) into factors with real coefficients.

   (a) $x^2 + 3x + 2$ (b) $3x^2 - 4x + 1$ (c) $x^3 - 7x$

   (d) $x^4 - 16$ (e) $x^3 - x^2 + 9x - 9$

2. Calculate the following (you may want to factor the denominator).

   (a) $\displaystyle\int \frac{1}{x^2 - 4x + 4}\, dx$ (b) $\displaystyle\int \frac{x+2}{x^2 + 5x + 6}\, dx$

   (c) $\displaystyle\int \frac{2x-1}{x^2+1}\, dx$ (d) $\displaystyle\int \frac{x^3 + 2x + 1}{x - 1}\, dx$

(1a) $x^2 + 3x + 2 = (x + 1)(x + 2)$

(1b) $3x^2 - 4x + 1 = (3x - 1)(x - 1)$

(1c) $x^3 - 7x = x(x + \sqrt{7})(x - \sqrt{7})$

(1d) $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4).$

(1e) $x^3 - x^2 - 9x + 9 = x^2(x - 1) - 9(x - 1)$
$= (x^2 - 9)(x - 1) = (x + 3)(x - 3)(x - 1)$

(2a) $\int \frac{1}{x^2-4x+4}\, dx = \int \frac{1}{(x-2)^2}\, dx = -(x - 2)^{-1} + C$

(2b) $\int \frac{x+2}{x^2+5x+6}\, dx = \int \frac{x+2}{(x+2)(x+3)}\, dx = \int \frac{1}{x+3}\, dx = \ln|x + 3| + C$

(2c) $\int \frac{2x-1}{x^2+1}\, dx = \int \frac{2x}{x^2+1} - \frac{1}{x^2+1}\, dx = \ln|x^2 + 1| - \tan^{-1}(x) + C$

(2d) $\int \frac{x^3+2x+1}{x-1}\, dx = \int x^2 + x + 3 + \frac{4}{x-1}\, dx =$
$\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + 4\ln|x - 1| + C.$

Reviewing polynomial long division

Calculate $\frac{x^3+2x+1}{x-1}$:

x into x^3 ↓ , x into x^2 ↓ , x into $3x$ ↓

$$\begin{array}{r|rrrrl}
 & x^2 & +\ x & +\ 3 & +\frac{4}{x-1} & \\ \hline
x-1 & x^3 & +\ 0 & +\ 2x & +\ 1 & \\
 & -(x^3 & -\ x^2) & & & \longleftarrow (x-1)x^2 \\ \hline
 & & x^2 & +\ 2x & +\ 1 & \\
 & & -(x^2 & -\ x) & & \longleftarrow (x-1)x \\ \hline
 & & & 3x & +\ 1 & \\
 & & & -(3x & -\ 3) & \longleftarrow (x-1)3 \\ \hline
 & & & & 4 & \longleftarrow \text{remainder}
\end{array}$$

So

$$\frac{x^3 + 2x + 1}{x - 1} = x^2 + x + 3 + \frac{4}{x - 1}.$$

Strategies for integrating rational functions

1. To compute
$$\int \frac{x+2}{x^2+5x+6}\,dx$$
we noted that
$$\frac{x+2}{x^2+5x+6} = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$
so that
$$\int \frac{x+2}{x^2+5x+6}\,dx = \int \frac{1}{x+3}\,dx = \ln|x+3| + C.$$

2. To compute
$$\int \frac{2x+5}{x^2+5x+6}\,dx,$$
we could note that
if $u = x^2+5x+6$, them $du = 2x+5$, so
$$\int \frac{2x+5}{x^2+5x+6}\,dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|x^2+5x+6|+C.$$

Strategies for integrating rational functions

3. How about
$$\int \frac{3x+13}{x^2+5x+6}\,dx?$$
What if I pointed out to you that
$$\frac{7}{x+2} - \frac{4}{x+3} = \frac{7(x+3)}{(x+2)(x+3)} - \frac{4(x+2)}{(x+2)(x+3)}$$
$$= \frac{7x+21-4x-8}{(x+2)(x+3)} = \frac{3x+13}{x^2+5x+6}?$$
Well, then
$$\int \frac{3x+13}{x^2+5x+6}\,dx = \int \frac{7}{x+2} - \frac{4}{x+3}\,dx$$
$$= 7\ln|x+2| - 4\ln|x+3| + C.$$

Strategies for integrating rational functions

How could I have found that

$$\frac{3x+13}{x^2+5x+6} = \frac{7}{x+2} - \frac{4}{x+3}$$

if I didn't already know?
Well, since the denominator factors as

$$x^2+5x+6 = (x+2)(x+3),$$

if the fraction can be written as the sum of two fractions with linear denominators, those denominators will be $(x+2)$ and $(x+3)$. So, for *some* A and B, we know

$$\frac{3x+13}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}.$$

So we can solve for A and B as follows!

Suppose A and B satisfy

$$\frac{3x+13}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

Then multiplying both sides by $(x+2)(x+3)$, we get

$$3x+13 = A(x+3) + B(x+2) = (A+B)x + (3A+2B).$$

Comparing both sides, we know that the constant terms have to match and the coefficients on x have to match (that's what it means for two polynomials to be equal!). So

$$3 = A+B \quad \text{and} \quad 13 = 3A+2B.$$

Plugging $A = 3-B$ (from the first equation) into the second equation, we get

$$13 = 3(3-B) + 2B = 9 - 3B + 2B = 9 - B.$$

So $\boxed{B=-4}$, and so $\boxed{A = 3-(-4) = 7}$. Thus

$$\frac{3x+13}{(x+2)(x+3)} = \frac{7}{x+2} + \frac{-4}{x+3}, \text{ as expected!}$$

You try

1. Solve for A and B such that

$$\frac{4x+11}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}.$$

2. Use your previous answer to calculate $\int \frac{4x+11}{(x-1)(x+4)}\ dx$.

You try

1. Solve for A and B such that
$$\frac{4x+11}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}.$$
Multiply both sides by $(x-1)(x+4)$ to get
$4x + 11 = A(x+4) + B(x-1) = (A+B)x + (4A - B)$.
Thus $4 = A + B$, so that $B = 4 - A$, and
$11 = 4A - B = 4A - (4 - A) = 5A - 4$. So
$\boxed{A = 3}$ and $\boxed{B = 1}$.

2. Use your previous answer to calculate $\int \frac{4x+11}{(x-1)(x+4)}\, dx$.
$$\int \frac{4x+11}{(x-1)(x+4)}\, dx = \int \frac{3}{x-1} + \frac{1}{x+4}\, dx$$
$$= \boxed{3\ln|x-1| + \ln|x+4|.}$$

Rules for splitting rational functions

If you're trying to split a rational function $Q(x)/P(x)$:

0. If $\deg(Q(x)) \geq \deg(P(x))$, use long division first.
1. Factor the denominator $P(x)$ (as far as possible) into degree 1 and 2 factors with real coefficients:

$$P(x) = p_1(x)p_2(x)\cdots p_k(x).$$

2. For each factor $p_i(x)$, add a term according to the following rules. Note that $p_i(x)$ might appear multiple times.
   (a) If p_i appears exactly once in the factorization:
   If p_i is degree 1, add a term of the form $A/p_i(x)$;
   If p_i is degree 2, add aa term of the form $(Ax+B)/p_i(x)$.
   Basically, add a term with $p_i(x)$ in the denominator and a generic polynomial of one degree less in the numerator.

For example, we would write

$$\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}, \quad \frac{1}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3},$$

$$\text{and} \quad \frac{1}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

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Basically, add a term with $p_i(x)$ in the denominator and a generic polynomial of one degree less in the numerator.

(b) If p_i appears exactly n times in the factorization:
If p_i is degree 1, add terms $A_j/(p_i(x))^j$ for $j = 1, 2, \ldots, n$;
If p_i is degree 2, add terms $(A_jx + B_j)/(p_i(x))^j$ for $j = 1, 2, \ldots, n$.

For example, (split doesn't depend on numerator!)

$$\frac{x^3+9x+15}{(x+2)(x^2+x+1)^2(x+1)^4}$$
$$= \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$
$$+ \frac{F}{x+1} + \frac{G}{(x+1)^2} + \frac{H}{(x+1)^3} + \frac{I}{(x+1)^4}.$$

You try:

Split the following rational functions. Don't solve for the constants; just set it up. If need be, factor the denominator first.

1. $\dfrac{1}{(x+3)(x-5)}$
2. $\dfrac{x}{3x^2-4x+1}$
3. $\dfrac{2x+1}{x^4-16}$
4. $\dfrac{10x^2-3x+1}{x^3(x+1)(x^2+3)^5}$

You try:

Split the following rational functions. Don't solve for the constants; just set it up. If need be, factor the denominator first.

1. $\dfrac{1}{(x+3)(x-5)} = \dfrac{A}{x+3} + \dfrac{B}{x-5}$
2. $\dfrac{x}{3x^2-4x+1} = \dfrac{A}{3x-1} + \dfrac{B}{x-1}$
3. $\dfrac{2x+1}{x^4-16} = \dfrac{A}{x+2} + \dfrac{B}{x-2} + \dfrac{Cx+D}{x^2+4}$
4. $\dfrac{10x^2-3x+1}{x^3(x+1)(x^2+3)^5}$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$+ \frac{E}{x^2+3} + \frac{Fx+G}{(x^2+3)^2} + \frac{Hx+I}{(x^2+3)^3} + \frac{Jx+K}{(x^2+3)^4} + \frac{Lx+M}{(x^2+3)^5}$$

Definition: This split form is called partial fractions decomposition.

You try:

Compute

$$\int \frac{x}{3x^2 - 4x + 1}\ dx.$$

(use the previous slide to split the fraction, solve for the unknowns, and use the split form to integrate)

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Soln: Write

$$\frac{x}{3x^2 - 4x + 1} = \frac{A}{3x - 1} + \frac{B}{x - 1},$$

so that

$$x = A(x-1) + B(3x-1) = (A + 3B)x + (-A - B).$$

Then $1 = A + 3B$ and $0 = -A - B$. So $1 = A + 3(\underbrace{-A}_{B}) = -2A$. Thus $A = -1/2$ and $B = 1/2$.

So

$$\int \frac{x}{3x^2 - 4x + 1}\ dx = \int \frac{-1/2}{3x - 1} + \frac{1/2}{x - 1}\ dx$$
$$= -\tfrac{1}{2}\ln|3x - 1| + \tfrac{1}{2}\ln|x - 1| + C.$$

You try:

Compute

$$\int \frac{3x^2 - x + 1}{x^3 + x}\, dx.$$

(factor the denominator, split the fraction, solve for the unknowns, and use the split form to integrate)

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Soln: Write

$$\frac{3x^2 - x + 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1},$$

so that

$$3x^2 - x + 1 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A.$$

Then $3 = (A + B)$, $-1 = C$, and $1 = A$. Thus $B = 2$.

So

$$\int \frac{3x^2 - x + 1}{x^3 + x}\ dx = \int \frac{1}{x} + \frac{2x - 1}{x^2 + 1}\ dx$$

$$= \ln|x| + \int \frac{2x}{x^2 + 1} - \frac{1}{x^2 + 1}\ dx = \ln|x| + \ln|x^2 + 1| - \tan^{-1}(x) + C$$

We could have been a little more clever with

$$\int \frac{3x^2 - x + 1}{x^3 + x}\ dx :$$

Notice

$$\frac{3x^2 - x + 1}{x^3 + x} = \frac{3x^2 + 1}{x^3 + x} - \frac{x}{x^3 + x}$$

$$= \frac{\frac{d}{dx}(x^3 + x)}{x^3 + x} - \frac{1}{x^2 + 1}.$$

So

$$\int \frac{3x^2 - x + 1}{x^3 + x}\ dx = \int \frac{3x^2 + 1}{x^3 + x}\ dx - \int \frac{1}{x^2 + 1}\ dx$$

$$= \ln|x^3 + x| - \tan^{-1}(x) + C.$$

What we got before:

$$\ln|x| + \ln|x^2 + 1| - \tan^{-1}(x) + C = \ln|x(x^2 + 1)| - \tan^{-1}(x) + C \checkmark$$

How the integration of each part goes

Suppose you've done a partial fractions decomposition. Then each part looks like

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^i}$$

(where i may be 1). For calculating

$$\int \frac{A}{(ax+b)^i}\,dx, \quad \text{let } u = ax+b \text{ every time.}$$

For calculating

$$\int \frac{Ax+B}{(ax^2+bx+c)^i}\,dx, \quad \text{there's a little more to be done.}$$

Whenever $i=1$, you will split this into two fractions:
one which has some constant times $\frac{d}{dx}(ax^2+bx+c) = 2ax+b$ in the numerator (so $u = ax^2+bx+c$), and one which has a constant in the numerator (so I can use $\frac{d}{dx}\tan^{-1}(\theta) = 1/(\theta^2+1)$, possibly after completing the square).

Integrating $(Ax+B)/(ax^2+bx+c)$

Example: $\displaystyle\int \frac{x+1}{x^2+1}\,dx.$

As I said, I want to break this up into two fractions, one which has some constant times $\frac{d}{dx}(x^2+1) = 2x$ in the numerator (so $u = x^2+1$), and one which has a constant in the numerator (so I can use $\frac{d}{dx}\tan^{-1}(x) = 1/(x^2+1)$:

$$\frac{x+1}{x^2+1} = \frac{x}{x^2+1} + \frac{1}{x^2+1},$$

so that

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \tfrac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C.$$

Integrating $(Ax+B)/(ax^2+bx+c)$

Example: $\displaystyle\int \frac{3x+5}{x^2+4}\,dx.$

Again, I want one $u = x^2+4$ (so $du = 2x$) and one $\tan^{-1}(u)$ integral.

$$\frac{3x+5}{x^2+4} = \frac{3x}{x^2+4} + \frac{5}{x^2+4} = \left(\frac{3}{2}\right)\frac{2x}{x^2+4} + \left(\frac{5}{4}\right)\frac{1}{(x/2)^2+1}.$$

So

$$\int \frac{3x+5}{x^2+4}\,dx = \frac{3}{2}\int \frac{2x}{x^2+4}\,dx + \frac{5}{4}\int \frac{1}{(x/2)^2+1}\,dx.$$

For the first, let $u = x^2+4$, so $du = 2x$, and thus

$$\left(\frac{3}{2}\right)\int \frac{2x}{x^2+4}\,dx = \frac{3}{2}\int u^{-1}\,du = \frac{3}{2}\ln|x^2+4| + C;$$

for the second, let $u = x/2$, so $2du = dx$, and thus

$$\frac{5}{4}\int \frac{1}{(x/2)^2+1}\,dx = \frac{5}{2}\int \frac{1}{u^2+1}\,du = \frac{5}{2}\tan^{-1}(x/2) + C.$$

Integrating $(Ax+B)/(ax^2+bx+c)$

Example: $\displaystyle\int \frac{x}{x^2+4x+5}\,dx.$

Complete the square of the denominator to get this fraction into the form of the previous two examples:

$$x^2+4x+7 = (x+2)^2 - 4 + 5 = (x+2)^2 + 1.$$

Now let $u = x+2$ (so that $du = dx$ and $x = u-2$):

$$\int \frac{x}{x^2+4x+5}\,dx = \int \frac{x}{(x+2)^2+1}\,dx = \int \frac{u-2}{u^2+1}\,du.$$

Now proceed as before!

$$\int \frac{u-2}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du - 2\int \frac{1}{u^2+1}\,du$$

$$= \tfrac{1}{2}\ln|u^2+1| - 2\tan^{-1}(u) + C = \tfrac{1}{2}\ln|(x+2)^2+1| - 2\tan^{-1}(x+2) + C.$$

Integrating $(Ax+B)/(ax^2+bx+c)^i$

If there's nothing obvious to be done, even for $i > 1$, complete the square of the denominator. In general, this goes as follows:

1. Factor out a^i: rewrite the denominator as $a^i(x^2+\beta x+\gamma)^i$, where $\beta = b/a$ and $\gamma = c/a$.
2. Complete the square of the rest:

$$x^2+\beta x+\gamma = (x+\beta/2)^2 + (\gamma - \beta^2/4) = (x+\beta/2)^2 + \alpha^2$$

   where $\alpha = \sqrt{\gamma - \beta^2/4}$. (real since the denom.'s not factorable)

3. Make the constant term 1 by factoring out α^2:

$$(ax^2+bx+c)^i = a^i((x+\beta/2)^2+\alpha^2)^i = a^i\alpha^{2i}\left(\left(\frac{x+\beta/2}{\alpha}\right)^2+1\right)^i.$$

4. Let $u = \frac{x+\beta/2}{\alpha}$. Use $x = \alpha x - \beta/2$ to rewrite the numerator.

(Don't try to memorize these equations. Just know that the process works every time.)

You try:

Compute

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$$

[Hint:

1. The degree of the numerator is not less than the degree of the denominator. So reduce first. You should end up with something that lookes like $A + \frac{Bx+D}{4x^2-4x+3}$.
2. Since the denom. is not factorable, get it into the form $k((f(x))^2+1)$ as on the previous slide. Then let $u = f(x)$.]

(See example 5 in section 6.3 for solution)