Today: 6.3 Partial fractions

Warm up: Recall that if you require *real* coefficients (not complex), polynomials always factor into degree 1 or 2 parts. For example,

$$x^3+x^2+3x-5=\underbrace{(x-1)(x+1-2i)(x+1+2i)}_{\text{allowing complex coeffs}}=\underbrace{(x-1)(x^2+2x+5)}_{\text{using only real coeffs}}$$

1. Factor the following polynomials (as far as possible) into factors with real coefficients.

   (a) x^2+3x+2 (b) $3x^2-4x+1$ (c) x^3-7x

   (d) x^4-16 (e) x^3-x^2+9x-9

2. Calculate the following (you may want to factor the denominator).

   (a) $\int \frac{1}{x^2-4x+4}\,dx$ (b) $\int \frac{x+2}{x^2+5x+6}\,dx$

   (c) $\int \frac{2x-1}{x^2+1}\,dx$ (d) $\int \frac{x^3+2x+1}{x-1}\,dx$

(1a) $x^2 + 3x + 2 = (x + 1)(x + 2)$

(1b) $3x^2 - 4x + 1 = (3x - 1)(x - 1)$

(1c) $x^3 - 7x = x(x + \sqrt{7})(x - \sqrt{7})$

(1d) $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$.

(1e) $$\begin{aligned} x^3 - x^2 - 9x + 9 &= x^2(x - 1) - 9(x - 1) \\ &= (x^2 - 9)(x - 1) = (x + 3)(x - 3)(x - 1) \end{aligned}$$

(2a) $\int \frac{1}{x^2-4x+4}\ dx = \int \frac{1}{(x-2)^2}\ dx = -(x - 2)^{-1} + C$

(2b) $\int \frac{x+2}{x^2+5x+6}\ dx = \int \frac{x+2}{(x+2)(x+3)}\ dx = \int \frac{1}{x+3}\ dx = \ln|x + 3| + C$

(2c) $\int \frac{2x-1}{x^2+1}\ dx = \int \frac{2x}{x^2+1} - \frac{1}{x^2+1}\ dx = \ln|x^2 + 1| - \tan^{-1}(x) + C$

(2d) $\int \frac{x^3+2x+1}{x-1}\ dx = \int x^2 + x + 3 + \frac{4}{x-1}\ dx =$
$\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + 4\ln|x - 1| + C.$

Reviewing polynomial long division

Calculate $\frac{x^3+2x+1}{x-1}$:

$$x - 1 \,\Big|\overline{\; x^3 \;+\; 0 \;+\; 2x \;+\; 1 \;}$$

Reviewing polynomial long division

Calculate $\frac{x^3+2x+1}{x-1}$:

$$\begin{array}{rl} & x \text{ into } x^3 \\ & \quad \downarrow \\ & \quad x^2 \\ x-1 & \big)\overline{\ x^3 + \ 0 \ + 2x + \ 1 \ } \end{array}$$

Reviewing polynomial long division

Calculate $\frac{x^3+2x+1}{x-1}$:

x into x^3

$$\begin{array}{r|l} & x^2 \\ \cline{2-2} x-1 & x^3 + 0 + 2x + 1 \\ & x^3 - x^2 \qquad \longleftarrow \; (x-1)x^2 \end{array}$$

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 & \quad \downarrow \\
 & \quad x^2 \\ \cline{2-2}
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\end{array}$$

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 & \quad x \text{ into } x^3 \\
 & \quad\;\; \downarrow \\
 & \;\; x^2 \\ \hline
x-1 & \;\; x^3 + 0 + 2x + 1 \\
 & -(x^3 - x^2) \qquad \longleftarrow (x-1)x^2 \\ \hline
 & \qquad\quad\;\; x^2 + 2x + 1
\end{array}$$

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Calculate $\frac{x^3+2x+1}{x-1}$:

x into x^3

x into x^2

$$\begin{array}{r l l}
 & \quad x^2 + x & \\
x-1 & \overline{\big)\; x^3 + 0 + 2x + 1} & \\
 & \underline{-(x^3 - x^2)\qquad\qquad} & \longleftarrow (x-1)x^2 \\
 & \qquad\quad x^2 + 2x + 1 &
\end{array}$$

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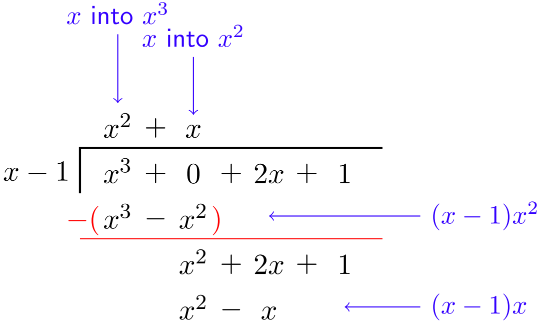
Calculate $\frac{x^3+2x+1}{x-1}$:



$$
\begin{array}{r|l}
 & \overset{\overset{x \text{ into } x^3}{\downarrow}}{x^2} + \overset{\overset{x \text{ into } x^2}{\downarrow}}{x} \\ \hline
x-1 & x^3 + 0 + 2x + 1 \\
 & -(x^3 - x^2) \qquad \longleftarrow (x-1)x^2 \\ \cline{2-2}
 & \qquad x^2 + 2x + 1 \\
 & \qquad x^2 - x \qquad \longleftarrow (x-1)x
\end{array}
$$

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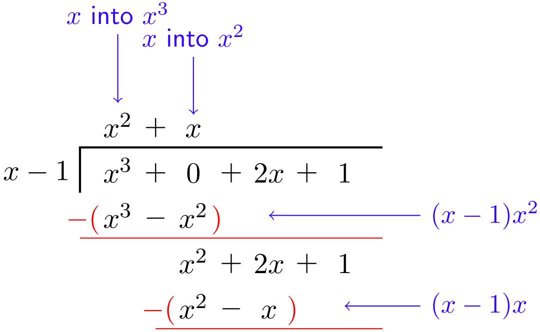
Calculate $\frac{x^3+2x+1}{x-1}$:



x into x^3

x into x^2

$$\begin{array}{r|l} & \quad x^2 + x \\ \hline x-1 & \quad x^3 + 0 + 2x + 1 \\ & -(x^3 - x^2) \longleftarrow (x-1)x^2 \\ \hline & \qquad\quad x^2 + 2x + 1 \\ & \qquad -(x^2 - x) \longleftarrow (x-1)x \\ \hline \end{array}$$

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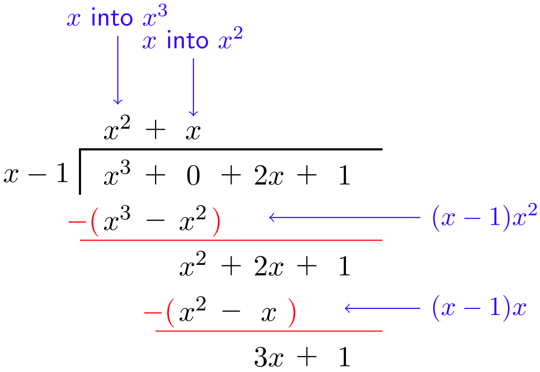
Calculate $\frac{x^3+2x+1}{x-1}$:



x into x^3

x into x^2

$$\begin{array}{r|lllllllll}
 & x^2 & + & x & & & & & & \\ \cline{2-8}
x-1 & x^3 & + & 0 & + & 2x & + & 1 & & \\
 & -(x^3 & - & x^2) & & & & & \longleftarrow & (x-1)x^2 \\ \cline{2-8}
 & & & x^2 & + & 2x & + & 1 & & \\
 & & & -(x^2 & - & x) & & & \longleftarrow & (x-1)x \\ \cline{4-8}
 & & & & & 3x & + & 1 & &
\end{array}$$

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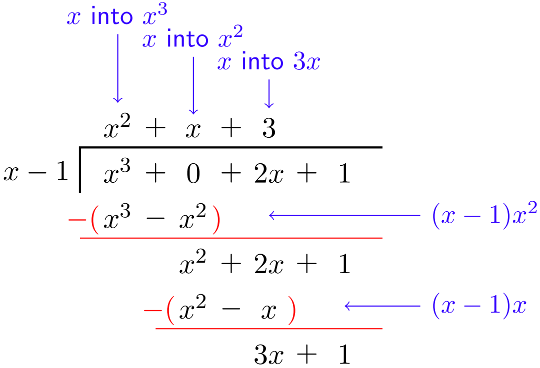

x into x^3

x into x^2

x into $3x$

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & x^2 + x + 3 \\ \cline{2-2}
x-1 & x^3 + 0 + 2x + 1 \\
\multicolumn{1}{r}{} & -(x^3 - x^2) \longleftarrow (x-1)x^2 \\ \cline{2-2}
\multicolumn{1}{r}{} & \qquad x^2 + 2x + 1 \\
\multicolumn{1}{r}{} & \qquad -(x^2 - x) \longleftarrow (x-1)x \\ \cline{2-2}
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\end{array}
$$

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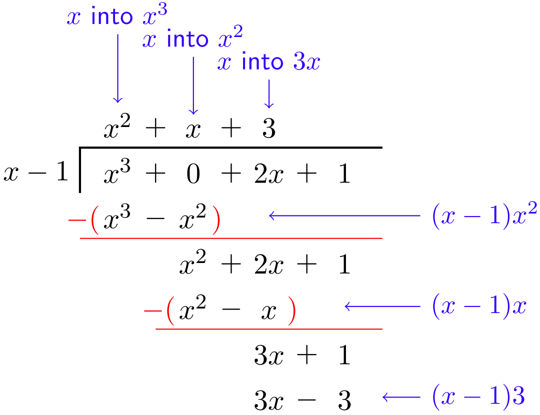

x into x^3

x into x^2

x into $3x$

$$
\begin{array}{r|lr}
 & \multicolumn{1}{l}{\quad x^2 + x + 3} & \\
\cline{2-2}
x-1 & x^3 + 0 + 2x + 1 & \\
 & -(x^3 - x^2) & \longleftarrow (x-1)x^2 \\
\cline{2-2}
 & \quad x^2 + 2x + 1 & \\
 & \quad -(x^2 - x) & \longleftarrow (x-1)x \\
\cline{2-2}
 & \qquad 3x + 1 & \\
 & \qquad 3x - 3 & \longleftarrow (x-1)3
\end{array}
$$

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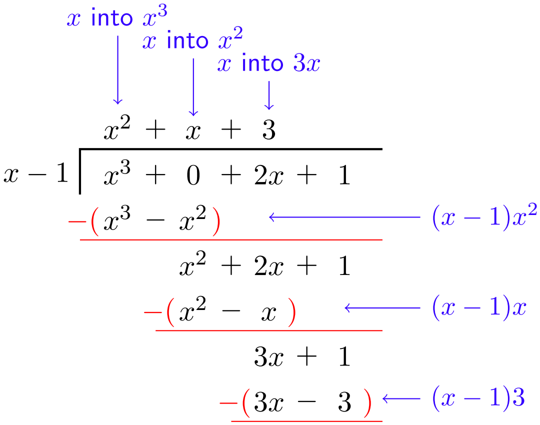

x into x^3 $\downarrow$

x into x^2 $\downarrow$

x into $3x$ $\downarrow$

$$
\begin{array}{rlll}
 & x^2 + x + 3 & & \\ \cline{2-2}
x-1 & \big|\; x^3 + 0 + 2x + 1 & & \\
 & -(x^3 - x^2) & \longleftarrow & (x-1)x^2 \\ \cline{2-2}
 & \quad x^2 + 2x + 1 & & \\
 & \quad -(x^2 - x) & \longleftarrow & (x-1)x \\ \cline{2-2}
 & \quad\quad 3x + 1 & & \\
 & \quad\quad -(3x - 3) & \longleftarrow & (x-1)3 \\ \cline{2-2}
\end{array}
$$

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$$
\begin{array}{rcccccccl}
 & x \text{ into } x^3 & & & & & & & \\
 & & & x \text{ into } x^2 & & & & & \\
 & & & & & x \text{ into } 3x & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & x^2 & + & x & + & 3 & & & \\ \cline{2-8}
x-1 \,\big) & x^3 & + & 0 & + & 2x & + & 1 & \\
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 & & & x^2 & + & 2x & + & 1 & \\
 & & & -(x^2 & - & x) & & & \longleftarrow (x-1)x \\ \cline{4-8}
 & & & & & 3x & + & 1 & \\
 & & & & & -(3x & - & 3) & \longleftarrow (x-1)3 \\ \cline{6-8}
 & & & & & & & 4 & \longleftarrow \text{remainder}
\end{array}
$$

Reviewing polynomial long division

Calculate $\frac{x^3+2x+1}{x-1}$:

x into x^3

x into x^2

x into $3x$

$$
\begin{array}{r|rrrrrrrl}
\multicolumn{1}{r}{} & x^2 & + & x & + & 3 & + & \frac{4}{x-1} & \\ \cline{2-8}
x-1 & x^3 & + & 0 & + & 2x & + & 1 & \\
\multicolumn{1}{r}{} & -(x^3 & - & x^2) & & & & & \longleftarrow (x-1)x^2 \\ \cline{2-8}
\multicolumn{1}{r}{} & & & x^2 & + & 2x & + & 1 & \\
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\end{array}
$$

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Calculate $\frac{x^3+2x+1}{x-1}$:

$$
\begin{array}{r|cccccccl}
 & x\text{ into }x^3 & & & & & & & \\
 & \downarrow & & x\text{ into }x^2 & & & & & \\
 & \downarrow & & \downarrow & & x\text{ into }3x & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & x^2 & + & x & + & 3 & + & \frac{4}{x-1} & \\
\cline{2-8}
x-1 & x^3 & + & 0 & + & 2x & + & 1 & \\
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\cline{6-8}
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\end{array}
$$

So

$$\frac{x^3+2x+1}{x-1} = x^2 + x + 3 + \frac{4}{x-1}.$$

Strategies for integrating rational functions

1. To compute

   $$\int \frac{x+2}{x^2+5x+6}\ dx$$

   we noted that

   $$\frac{x+2}{x^2+5x+6} = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$

   so that

   $$\int \frac{x+2}{x^2+5x+6}\ dx = \int \frac{1}{x+3}\ dx = \ln|x+3| + C.$$

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Strategies for integrating rational functions

How could I have found that

$$\frac{3x+13}{x^2+5x+6} = \frac{7}{x+2} - \frac{4}{x+3}$$

if I didn't already know?

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if the fraction can be written as the sum of two fractions with linear denominators, those denominators will be $(x+2)$ and $(x+3)$.

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$$\frac{3x+13}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}.$$

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$$\frac{3x+13}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}.$$

So we can solve for A and B as follows!

Suppose A and B satisfy

$$\frac{3x+13}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

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Comparing both sides, we know that the constant terms have to match and the coefficients on x have to match (that's what it means for two polynomials to be equal!).

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Plugging $A = 3 - B$ (from the first equation) into the second equation, we get

$$13 = 3(3-B) + 2B$$

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Plugging $A = 3 - B$ (from the first equation) into the second equation, we get

$$13 = 3(3-B) + 2B = 9 - 3B + 2B$$

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So $\boxed{B = -4}$

Suppose A and B satisfy

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Plugging $A = 3 - B$ (from the first equation) into the second equation, we get

$$13 = 3(3-B) + 2B = 9 - 3B + 2B = 9 - B.$$

So $\boxed{B = -4}$, and so $\boxed{A = 3 - (-4) = 7}$.

Suppose A and B satisfy

$$\frac{3x+13}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

Then multiplying both sides by $(x+2)(x+3)$, we get

$$3x + 13 = A(x+3) + B(x+2) = (A+B)x + (3A+2B).$$

Comparing both sides, we know that the constant terms have to match and the coefficients on x have to match (that's what it means for two polynomials to be equal!). So

$$3 = A + B \quad \text{and} \quad 13 = 3A + 2B.$$

Plugging $A = 3 - B$ (from the first equation) into the second equation, we get

$$13 = 3(3-B) + 2B = 9 - 3B + 2B = 9 - B.$$

So $\boxed{B = -4}$, and so $\boxed{A = 3 - (-4) = 7}$. Thus

$$\frac{3x+13}{(x+2)(x+3)} = \frac{7}{x+2} + \frac{-4}{x+3}, \text{ as expected!}$$

You try

1. Solve for A and B such that
$$\frac{4x+11}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}.$$

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Multiply both sides by $(x-1)(x+4)$ to get
$4x+11 = A(x+4) + B(x-1) = (A+B)x + (4A-B)$.
Thus $4 = A+B$, so that $B = 4-A$, and
$11 = 4A - B = 4A - (4-A) = 5A - 4$. So
$\boxed{A=3}$ and $\boxed{B=1}$.

2. Use your previous answer to calculate $\int \frac{4x+11}{(x-1)(x+4)}\,dx$.

$$\int \frac{4x+11}{(x-1)(x+4)}\,dx = \int \frac{3}{x-1} + \frac{1}{x+4}\,dx$$

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$$= \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

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$$\frac{5x+2}{(x+2)(x^2+x+1)^2(x+1)^4}$$

$$= \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

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$$\frac{x^3+9x+15}{(x+2)(x^2+x+1)^2(x+1)^4}$$

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You try:

Split the following rational functions. Don't solve for the constants; just set it up. If need be, factor the denominator first.

1. $\frac{1}{(x+3)(x-5)}$
2. $\frac{x}{3x^2-4x+1}$
3. $\frac{2x+1}{x^4-16}$
4. $\frac{10x^2-3x+1}{x^3(x+1)(x^2+3)^5}$

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$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1}$$

$$+ \frac{E}{x^2+3} + \frac{Fx+G}{(x^2+3)^2} + \frac{Hx+I}{(x^2+3)^3} + \frac{Jx+K}{(x^2+3)^4} + \frac{Lx+M}{(x^2+3)^5}$$

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Definition: This split form is called partial fractions decomposition.

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(use the previous slide to split the fraction, solve for the unknowns, and use the split form to integrate)

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so that

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Then $1 = A + 3B$ and $0 = -A - B$.

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Then $1 = A + 3B$ and $0 = -A - B$. So
$1 = A + 3(\underbrace{-A}_{B}) = -2A$.

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So

$$\int \frac{x}{3x^2 - 4x + 1}\ dx = \int \frac{-1/2}{3x - 1} + \frac{1/2}{x - 1}\ dx$$
$$= -\tfrac{1}{2} \ln|3x - 1| + \tfrac{1}{2} \ln|x - 1| + C.$$

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so that

$$3x^2 - x + 1 = A(x^2 + 1) + (Bx + C)x$$

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$$\int \frac{3x^2 - x + 1}{x^3 + x}\ dx.$$

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$$\int \frac{3x^2 - x + 1}{x^3 + x} \; dx :$$

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$$\ln|x| + \ln|x^2 + 1| - \tan^{-1}(x) + C = \ln|x(x^2 + 1)| - \tan^{-1}(x) + C \ \checkmark$$

How the integration of each part goes

Suppose you've done a partial fractions decomposition. Then each part looks like

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Integrating $(Ax + B)/(ax^2 + bx + c)$

Example: $\displaystyle\int \frac{x+1}{x^2+1}\,dx.$

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Example: $\displaystyle\int \frac{3x+5}{x^2+4}\,dx.$

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Integrating $(Ax + B)/(ax^2 + bx + c)$

Example: $\displaystyle \int \frac{x}{x^2 + 4x + 5}\, dx.$

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Complete the square of the denominator to get this fraction into the form of the previous two examples:

Integrating $(Ax + B)/(ax^2 + bx + c)$

Example: $\displaystyle\int \frac{x}{x^2 + 4x + 5}\, dx.$

Complete the square of the denominator to get this fraction into the form of the previous two examples:

$$x^2 + 4x + 7 = (x + 2)^2 - 4 + 5$$

Integrating $(Ax+B)/(ax^2+bx+c)$

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Now proceed as before!

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Now proceed as before!

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$$=\tfrac{1}{2}\ln|u^2+1|-2\tan^{-1}(u)+C=\tfrac{1}{2}\ln|(x+2)^2+1|-2\tan^{-1}(x+2)+C.$$

Integrating $(Ax + B)/(ax^2 + bx + c)^i$

If there's nothing obvious to be done, even for $i > 1$, complete the square of the denominator.

Integrating $(Ax + B)/(ax^2 + bx + c)^i$

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(Don't try to memorize these equations. Just know that the process works every time.)

You try:

Compute

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\ dx.$$

[Hint:

1. The degree of the numerator is not less than the degree of the denominator. So reduce first. You should end up with something that lookes like $A + \frac{Bx+D}{4x^2-4x+3}$.
2. Since the denom. is not factorable, get it into the form $k((f(x))^2+1)$ as on the previous slide. Then let $u = f(x)$.]