Today: 6.2 Trig substitution

Warm up:

1. Calculate the following integrals.

$$\text{(a)} \int \cos^2(x)\ dx \quad \text{(b)} \int \cos^2(x)\sin^2(x)\ dx$$

$$\text{(c)} \int \cot^2(x)\ dx \quad \text{(d)} \int \tan^3(x)\ dx$$

2. Simplify the following expressions.

$$\text{(a)}\ \sin(\cos^{-1}(x)) \quad \text{(b)}\ \tan(\sec^{-1}(x))$$

$$\text{(c)}\ \sin(2\cos^{-1}(x)) \quad \text{(d)}\ \cos(2\cos^{-1}(x))$$

(1a) $\int \cos^2(x)\ dx = \frac{1}{2}\int 1+\cos(2x)\ dx$

$= \frac{1}{2}(x+\frac{1}{2}\sin(2x))+C.$

(1b) $\int \cos^2(x)\sin^2(x)\ dx = \int (\frac{1}{2}\sin(2x))^2\ dx$

$= \frac{1}{4}\int \sin^2(2x)\ dx = \frac{1}{8}\int 1-\cos(4x)\ dx$

$= \frac{1}{8}(x-\frac{1}{4}\sin(4x))+C = \frac{1}{32}(4x-\sin(4x))+C.$

(1c) $\int \cot^2(x)\ dx = \frac{1}{2}\int \csc^2(x)-1\ dx$

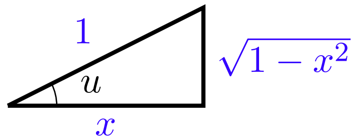
$= -\cot(x)-x+C$

(1d) $\int \tan^3(x)\ dx = \int \tan(x)(\sec^2(x)-1)\ dx$

$= \int \underbrace{\tan(x)\sec^2(x)\ dx}_{=u\ du,\text{ with } u=\tan(x)} - \int_{-u^{-1}} \underbrace{\tan(x)\ dx}_{du,\text{ with } u=\cos(x)}$

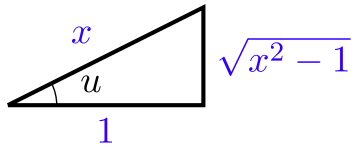
$\frac{1}{2}\tan^2(x)+\ln|\cos(x)|+C.$

(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus use the triangle



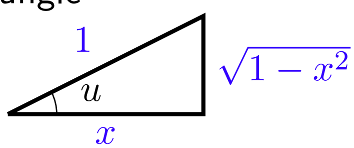
So $\sin(\cos^{-1}(x)) = \sin(u) = \boxed{\sqrt{1-x^2}}$.

(2b) $\tan(\sec^{-1}(x))$: Let $u = \sec^{-1}(x)$ so that $\sec(u) = x$. Thus use the triangle



So $\tan(\sec^{-1}(x)) = \tan(u) = \boxed{\sqrt{1-x^2}}$.

(2c) $\sin(2\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus again use the triangle



Thus

$$\sin(2\cos^{-1}(x)) = \sin(2u) = 2\sin(u)\cos(u) = \boxed{2x\sqrt{1-x^2}}.$$

(2d) $\cos(2\cos^{-1}(x))$: Using the same substitution and triangle as above, we have

$$\cos(2\cos^{-1}(x)) = \cos(2u) = (\cos(u))^2 - (\sin(u))^2$$

$$= x^2 - (1-x^2) = \boxed{2x^2-1}.$$

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\;dx.$$

Let $u = 1 - x^2$. Then $du = -2x\;dx$. So

$$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int u^{1/2}\,dx = -\tfrac{1}{2}\tfrac{2}{3}u^{3/2}+C = -\tfrac{1}{3}(1-x^2)^{3/2}+C.$$

A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\;dx.$$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\;dx = \frac{1}{2u}dx$, and thus $dx = 2u\;du$. So

$$\int e^{\sqrt{x}}\;dx = \int e^u(2u)\;du = 2ue^u - 2\int e^u\;du$$

(integration by parts with $f(u) = 2u$ and $g'(u) = e^u$)

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

Trig substitution, or "reverse u-sub"

How about

$$\int \sqrt{1-x^2}\;dx?$$

We could try letting $u = 1 - x^2$, so that $x = \sqrt{1-u}$. Further, $du = -2x\;dx = -2\sqrt{1-u}\;du$. So

$$\int \sqrt{1-x^2}\;dx = -\tfrac{1}{2}\int \frac{\sqrt{u}}{\sqrt{1-u}}\;du = -\tfrac{1}{2}\int \sqrt{\frac{u}{1-u}}\;du\ldots$$

Trig substitution, or "reverse u-sub"

How about

$$\int \sqrt{1-x^2}\ dx?$$

Instead of using a substitution that looks like $u = f(x)$, we can try making a substitution that looks like $x = f(u)$. Is there a function $f(u)$ such that $1 - f^2(u)$ is a perfect square? Think

$$\cos^2(u) + \sin^2(u) = 1.$$

Let $x = \cos(u)$. Then $dx = -\sin(u)\ du$. So

$$\int \sqrt{1-x^2}\ dx = \int \sqrt{1-\cos^2(u)} \cdot (-\sin(u))\ du$$

$$= -\int \sqrt{\sin^2(u)} \cdot \sin(u)\ du = -\int \sin^2(u)\ du$$

$$= \tfrac{1}{2}\int \cos(2u) - 1\ du = \tfrac{1}{2}(\tfrac{1}{2}\sin(2u) - u) + C$$

$$= \tfrac{1}{4}\sin(2\cos^{-1}(x))) - \tfrac{1}{2}\cos^{-1}(x) + C.$$

Simplifying $\sin(2\sin^{-1}(x)))$

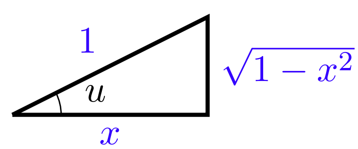
So we don't get too bogged down, let's go back to writing this as

$$\sin(2u),\ \text{where}\ \cos(u) = x\ \text{so that}\ u = \cos^{-1}(x).$$

First,

$$\sin(2u) = 2\sin(u)\cos(u).$$

Next, use the triangle



$$\sin(2u) = 2\sin(u)\cos(u) = 2\sqrt{1-x^2} \cdot x.$$

So

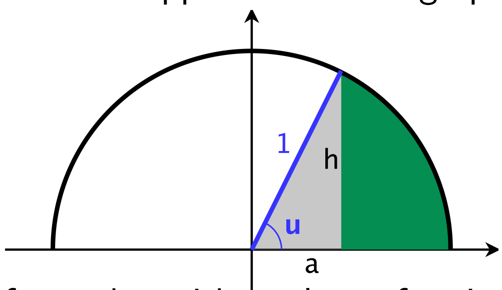
$$\int \sqrt{1-x^2}\ dx = \cdots = -\tfrac{1}{2}\cos^{-1}(x) + \tfrac{1}{4}\sin(2\sin^{-1}(x))) + C$$

$$= \tfrac{1}{2}x\sqrt{1-x^2} - \tfrac{1}{2}\cos^{-1}(x) + C$$

Check against geometry:

$$\boxed{\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right) + C}$$

Note: $y = \sqrt{1-x^2}$ is the upper half of the graph of $y^2 + x^2 = 1$:



Recall: the area of a wedge with angle u of a circle or radius r is

$$A = (u/2\pi)\pi r^2 = \tfrac{1}{2}ur^2.$$

So, for example, the integral $I = \int_a^1 \sqrt{1-x^2}\,dx$ should be

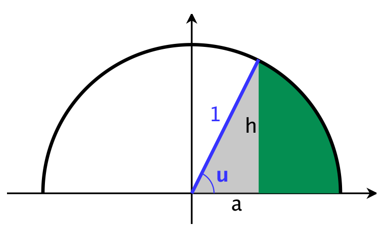
$$(\text{area of the wedge}) - (\text{area of the triangle}) = \tfrac{1}{2}u - \tfrac{1}{2}ah.$$

Since $h = \sqrt{1-a^2}$ and $u = \cos^{-1}(a)$, we have

$$I = \tfrac{1}{2}\cos^{-1}(a) - \tfrac{1}{2}a\sqrt{1-a^2}.$$

Check against geometry:

$$\boxed{\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right) + C}$$



Geometrically,

$$\int_a^1 \sqrt{1-x^2}\,dx = \tfrac{1}{2}u - \tfrac{1}{2}ah = \tfrac{1}{2}\cos^{-1}(a) - \tfrac{1}{2}a\sqrt{1-a^2}.$$

Checking against the formula we computed:

$$\int_a^1 \sqrt{1-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right)\Big|_{x=a}^{1}$$

$$= \tfrac{1}{2}\left((1\cdot 0 - \cos^{-1}(1)) - (a\sqrt{1-a^2} - \cos^{-1}(a))\right)$$

$$= \tfrac{1}{2}\cos^{-1}(a) - \tfrac{1}{2}a\sqrt{1-a^2} \quad \checkmark \quad (\cos^{-1}(1) = 0).$$

You try:

Calculate the following integrals using the suggested substitution.
Be sure to simplify your answers.

1. $\int \frac{\sqrt{1-x^2}}{x^2}\ dx$ using $x = \sin(u)$

2. $\int \frac{1}{x^2\sqrt{1+x^2}}\ dx$ using $x = \tan(u)$

3. $\int \frac{x}{\sqrt{1+x^2}}\ dx$ two ways:
   (a) Let $x = \tan(u)$ (b) Let $u = 1 + x^2$.

You try:

Calculate the following integrals using the suggested substitution. Be sure to simplify your answers.

1. $\int \frac{\sqrt{1-x^2}}{x^2}\ dx$ using $x = \sin(u)$: $dx = \cos(u)\ du$, so
   $I = \int \frac{\sqrt{1-\sin^2(u)}}{\sin^2(u)} \cdot \cos(u)\ du = \int \frac{\cos^2(x)}{\sin^2(u)}\ du$
   $= \int \cot^2(u)\ du = -\cot(u) - u + C$
   $= -\cot(\sin^{-1}(x)) - \sin^{-1}(x) + C = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}(x) + C$
2. $\int \frac{1}{x^2\sqrt{1+x^2}}\ dx$ using $x = \tan(u)$: $dx = \sec^2(u)\ du$, so
   $I = \int \frac{1}{\tan^2(u)\sqrt{1+\tan^2(u)}}\ \sec^2(u)\ du = \int \frac{\sec^2(u)}{\tan^2(u)\sec(u)}\ du$
   $= \int \frac{\cos(u)}{\sin^2(u)}\ du = -\sin^{-1}(u) + C = -\csc(\tan^{-1}(x)) + C$
   $= -\frac{\sqrt{1+x^2}}{x} + C.$
3. $\int \frac{x}{\sqrt{1+x^2}}\ dx$ two ways:
   (a) Let $x = \tan(u)$ (b) Let $u = 1 + x^2$.

$$= \sqrt{1+x^2} + C$$

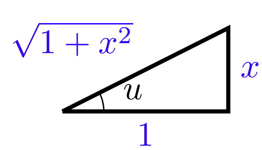
You try:

Calculate the following integrals using the suggested substitution. Be sure to simplify your answers.

3. $\int \frac{x}{\sqrt{1+x^2}}\ dx$ two ways:
   (a) Let $x = \tan(u)$: $dx = \sec^2(u)\ du$, so that

$$\int \frac{x}{\sqrt{1+x^2}}\ dx = \int \frac{\tan(u)}{\sqrt{1+\tan^2(x)}}\ \sec^2(u)\ du$$

$$= \int \sec(u)\tan(u)\ du = \sec(u) + C = \sqrt{1+x^2} + C$$



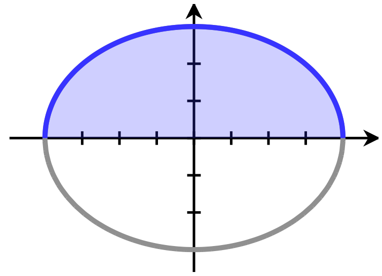
   (b) Let $u = 1 + x^2$: Then $du = 2x\ dx$, so that

$$\int \frac{x}{\sqrt{1+x^2}}\ dx = \frac{1}{2}\int u^{-1/2}\ du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1+x^2} + C.$$

Another geometric example

Compute the area of an ellipse with minor radius 3 and major radius 4.

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$



The desired area is half the area under the curve $y = 3\sqrt{1-(x/4)^2}$. So $A = 2 \cdot 3 \int_{-4}^{4} \sqrt{1-(x/4)^2}\, dx.$

Let $x/4 = \sin(u)$, so that $dx = 4\cos(u)\, du$. Thus,

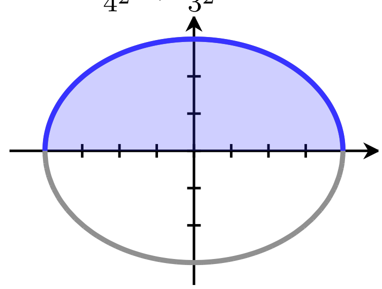
$$\int \sqrt{1 - x^2/4^2}\, dx = \int \sqrt{1 - \sin^2(u)}\, 4\cos(u)\, du = 4 \int \cos^2(u)\, du$$

$$= 4 \cdot \tfrac{1}{2}(u + \cos(u)\sin(u)) + C = 2\left(\sin^{-1}(x/4) + (x/4)\sqrt{1-(x/4)^2}\right) + C$$

Another geometric example

Compute the area of an ellipse with minor radius 3 and major radius 4.

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$



The desired area is half the area under the curve $y = 3\sqrt{1-(x/4)^2}$. So

$$A = 6 \int_{-4}^{4} \sqrt{1-(x/4)^2}\, dx$$

$$= 6\left(2\left(\sin^{-1}(x/4) + (x/4)\sqrt{1-(x/4)^2}\right)\right)\Big|_{x=-4}^{4}$$

$$= 12\left(\left(\sin^{-1}(1) + \sqrt{1-1}\right) - \left(\sin^{-1}(x-1) + (-1)\sqrt{1-1}\right)\right)$$

$$= 12(\pi/2 - (-\pi/2)) = \boxed{12\pi}.$$

Completing the square

Compute

$$\int \frac{1}{\sqrt{x^2 - 4x}}\ dx$$

Rewrite

$$x^2 - 4x = x^2 - 4x + 4 - 4 = (x-2)^2 - 4 = 4(\left(\tfrac{x-2}{2}\right)^2 - 1).$$

Recall, this is called completing the square. In general, if you're interested in rewriting

$$ax^2 + bx + c, \qquad \text{note that } (x + b/2a)^2 = x^2 + (b/a)x + (b/2a)^2.$$

So

$$ax^2+bx+c = a\left(x^2 + (b/a)x\right) + c = a\left((x+b/2a)^2 - (b/2a)^2\right) + c$$

$$= \left(\sqrt{a}(x+b/2a)^2\right) - \tfrac{1}{a}(b/2)^2 + c.$$

Completing the square

Compute

$$\int \frac{1}{\sqrt{x^2 - 4x}}\ dx$$

Rewrite

$$x^2 - 4x = x^2 - 4x + 4 - 4 = (x-2)^2 - 4.$$

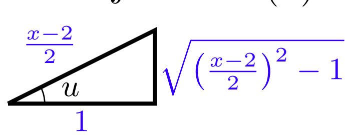
Then

$$\int \frac{1}{\sqrt{x^2-4x}}\ dx = \int \frac{1}{\sqrt{(x-2)^2-4}}\ dx = \int \frac{1}{2\sqrt{((x-2)/2)^2 - 1}}\ dx.$$

Let $(x-2)/2 = \sec(u)$. Then $\frac{1}{2}\ dx = \sec(u)\tan(u)\ du$, so that

$$I = \int \frac{1}{2\sqrt{\sec^2(u) - 1}}\ 2\sec(u)\tan(u)\ du$$

$$= \int \frac{\sec(u)\tan(u)}{\tan(u)}\ du = \ln|\sec(u) + \tan(u)| + C$$



$$= \ln\left|\tfrac{x-2}{2} + \sqrt{(\tfrac{x-2}{2})^2 - 1}\right| + C$$

You try:

Calculate the following integrals. Be sure to simplify your answers. Remember, the Pythagorean identities are

$$\cos^2(u) + \sin^2(u) = 1 \quad \text{and} \quad 1 + \tan^2(u) = \sec^2(u).$$

1. $\int \frac{1}{\sqrt{x^2-1}}\, dx$

2. $\int \sqrt{3 - x^2}\, dx$

3. $\int \frac{e^x}{e^{2x}\sqrt{1+e^{2x}}}\, dx$