Today: 6.2 Trig substitution

Warm up:

1. Calculate the following integrals.

(a) $\int \cos^2(x)\ dx$ (b) $\int \cos^2(x)\sin^2(x)\ dx$

(c) $\int \cot^2(x)\ dx$ (d) $\int \tan^3(x)\ dx$

2. Simplify the following expressions.

(a) $\sin(\cos^{-1}(x))$ (b) $\tan(\sec^{-1}(x))$

(c) $\sin(2\cos^{-1}(x))$ (d) $\cos(2\cos^{-1}(x))$

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$$\text{(c)} \int \cot^2(x)\ dx \quad \text{(d)} \int \tan^3(x)\ dx$$

2. Simplify the following expressions.

$$\text{(a)}\ \sin(\cos^{-1}(x)) \quad \text{(b)}\ \tan(\sec^{-1}(x))$$

$$\text{(c)}\ \sin(2\cos^{-1}(x)) \quad \text{(d)}\ \cos(2\cos^{-1}(x))$$

Answers:

$$\text{(1a)}\ \tfrac{1}{2}(x+\cos(x)\sin(x))+C \quad \text{(1b)}\ \tfrac{1}{32}(4x-\sin(4x))+C$$

$$\text{(1c)}\ -\cot(x)-x+C \quad \text{(1d)}\ \tfrac{1}{2}\tan^2(x)+\ln|\cos(x)|+C$$

$$\text{(2a)}\ \sqrt{1-x^2} \quad \text{(2b)}\ \sqrt{x^2-1} \quad \text{(2c)}\ 2x\sqrt{1-x^2} \quad \text{(2d)}\ 2x^2-1$$

(1a) $\int \cos^2(x)\ dx$

(1b) $\int \cos^2(x)\sin^2(x)\ dx$

(1c) $\int \cot^2(x)\ dx$

(1d) $\int \tan^3(x)\ dx$

(1a) $\int \cos^2(x)\ dx = \frac{1}{2} \int 1 + \cos(2x)\ dx$

(1b) $\int \cos^2(x) \sin^2(x)\ dx$

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(1b) $$\int \cos^2(x)\sin^2(x)\ dx = \int (\tfrac{1}{2}\sin(2x))^2\ dx$$

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(1b) $$\int \cos^2(x)\sin^2(x)\ dx = \int (\tfrac{1}{2}\sin(2x))^2\ dx$$
$$= \tfrac{1}{4}\int \sin^2(2x)\ dx$$

(1c) $$\int \cot^2(x)\ dx$$

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$$= \tfrac{1}{4}\int \sin^2(2x)\ dx = \tfrac{1}{8}\int 1 - \cos(4x)\ dx$$

(1c) $$\int \cot^2(x)\ dx$$

(1d) $$\int \tan^3(x)\ dx$$

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(1c) $$\int \cot^2(x)\ dx = \tfrac{1}{2}\int \csc^2(x)-1\ dx$$

(1d) $$\int \tan^3(x)\ dx$$

(1a) $$\int \cos^2(x)\ dx = \tfrac{1}{2}\int 1+\cos(2x)\ dx$$
$$= \tfrac{1}{2}(x + \tfrac{1}{2}\sin(2x)) + C.$$

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(1c) $$\int \cot^2(x)\ dx = \tfrac{1}{2}\int \csc^2(x) - 1\ dx$$
$$= -\cot(x) - x + C$$

(1d) $$\int \tan^3(x)\ dx$$

(1a) $$\int \cos^2(x)\ dx = \tfrac{1}{2}\int 1+\cos(2x)\ dx$$
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(1d) $$\int \tan^3(x)\ dx = \int \tan(x)(\sec^2(x)-1)\ dx$$

(1a)
$$\int \cos^2(x)\ dx = \tfrac{1}{2}\int 1+\cos(2x)\ dx$$
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(1d)
$$\int \tan^3(x)\ dx = \int \tan(x)(\sec^2(x)-1)\ dx$$
$$= \int \underbrace{\tan(x)\sec^2(x)\ dx}_{=u\ du,\ \text{with } u=\tan(x)} - \int_{-u^{-1}} \underbrace{\tan(x)\ dx}_{du,\ \text{with } u=\cos(x)}$$

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$$\tfrac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$$

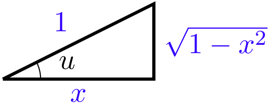
(2a) $\sin(\cos^{-1}(x))$:

(2b) $\tan(\sec^{-1}(x))$:

(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$.

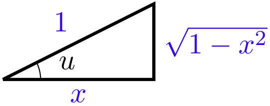
(2b) $\tan(\sec^{-1}(x))$:

(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus use the triangle



(2b) $\tan(\sec^{-1}(x))$:

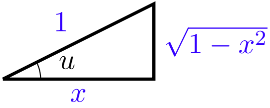
(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus use the triangle



So $\sin(\cos^{-1}(x)) = \sin(u) = \boxed{\sqrt{1-x^2}}$.

(2b) $\tan(\sec^{-1}(x))$:

(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus use the triangle



So $\sin(\cos^{-1}(x)) = \sin(u) = \boxed{\sqrt{1-x^2}}$.

(2b) $\tan(\sec^{-1}(x))$: Let $u = \sec^{-1}(x)$ so that $\sec(u) = x$.

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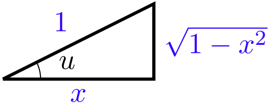
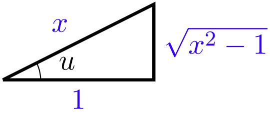
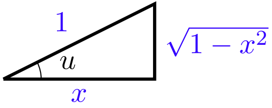


So $\sin(\cos^{-1}(x)) = \sin(u) = \boxed{\sqrt{1-x^2}}$.

(2b) $\tan(\sec^{-1}(x))$: Let $u = \sec^{-1}(x)$ so that $\sec(u) = x$. Thus use the triangle

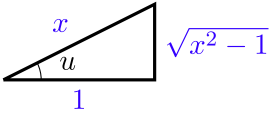


(2a) $\sin(\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus use the triangle



So $\sin(\cos^{-1}(x)) = \sin(u) = \boxed{\sqrt{1 - x^2}}$.

(2b) $\tan(\sec^{-1}(x))$: Let $u = \sec^{-1}(x)$ so that $\sec(u) = x$. Thus use the triangle



So $\tan(\sec^{-1}(x)) = \tan(u) = \boxed{\sqrt{1 - x^2}}$.

(2c) $\sin(2\cos^{-1}(x))$:

(2d) $\cos(2\cos^{-1}(x))$:

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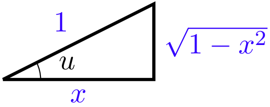

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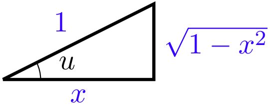
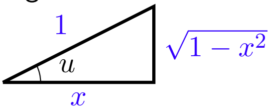


Thus

$$\sin(2\cos^{-1}(x)) = \sin(2u)$$

(2d) $\cos(2\cos^{-1}(x))$:

(2c) $\sin(2\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus again use the triangle

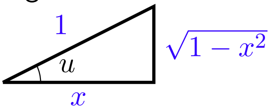


Thus

$$\sin(2\cos^{-1}(x)) = \sin(2u) = 2\sin(u)\cos(u)$$

(2d) $\cos(2\cos^{-1}(x))$:

(2c) $\sin(2\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus again use the triangle



Thus

$$\sin(2\cos^{-1}(x)) = \sin(2u) = 2\sin(u)\cos(u) = \boxed{2x\sqrt{1-x^2}}.$$

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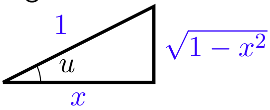

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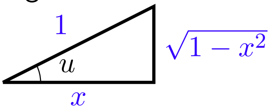
(2d) $\cos(2\cos^{-1}(x))$: Using the same substitution and triangle as above, we have

$$\cos(2\cos^{-1}(x)) = \cos(2u)$$

(2c) $\sin(2\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus again use the triangle



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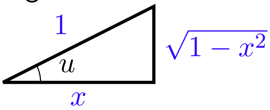
$$\sin(2\cos^{-1}(x)) = \sin(2u) = 2\sin(u)\cos(u) = \boxed{2x\sqrt{1-x^2}}.$$

(2d) $\cos(2\cos^{-1}(x))$: Using the same substitution and triangle as above, we have

$$\cos(2\cos^{-1}(x)) = \cos(2u) = (\cos(u))^2 - (\sin(u))^2$$

(2c) $\sin(2\cos^{-1}(x))$: Let $u = \cos^{-1}(x)$ so that $\cos(u) = x$. Thus again use the triangle



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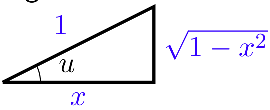

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(2d) $\cos(2\cos^{-1}(x))$: Using the same substitution and triangle as above, we have

$$\cos(2\cos^{-1}(x)) = \cos(2u) = (\cos(u))^2 - (\sin(u))^2$$

$$= x^2 - (1 - x^2) = \boxed{2x^2 - 1}.$$

Trig substitution, or “reverse u-sub”

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\;dx.$$

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\;dx.$$

Let $u = 1 - x^2$. Then $du = -2x\;dx$.

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\,dx.$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$. So

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,dx$$

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\ dx.$$

Let $u = 1 - x^2$. Then $du = -2x\ dx$. So

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,dx = -\frac{1}{2}\frac{2}{3}u^{3/2}+C$$

Trig substitution, or “reverse u-sub”

Normal straightforward u-sub:

Calculate

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A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\,dx.$$

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\,dx.$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$. So

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A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\,dx.$$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\,dx$

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\,dx.$$

Let $u = 1 - x^2$. Then $du = -2x\ dx$. So

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A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\ dx.$$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\ dx = \frac{1}{2u}dx$, and thus $dx = 2u\ du$.

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

$$\int x\sqrt{1-x^2}\;dx.$$

Let $u = 1 - x^2$. Then $du = -2x\;dx$. So

$$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int u^{1/2}\,dx = -\tfrac{1}{2}\tfrac{2}{3}u^{3/2}+C = -\tfrac{1}{3}(1-x^2)^{3/2}+C.$$

A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\;dx.$$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\;dx = \frac{1}{2u}dx$, and thus $dx = 2u\;du$. So

$$\int e^{\sqrt{x}}\;dx = \int e^{u}(2u)\;du$$

Trig substitution, or “reverse u-sub”

Normal straightforward u-sub:

Calculate

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Let $u = 1 - x^2$. Then $du = -2x\,dx$. So

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A little more sophistication: Calculate

$$\int e^{\sqrt{x}}\,dx.$$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}dx$, and thus $dx = 2u\,du$. So

$$\int e^{\sqrt{x}}\,dx = \int e^u(2u)\,du = 2ue^u - 2\int e^u\,du$$

(integration by parts with $f(u) = 2u$ and $g'(u) = e^u$)

Trig substitution, or "reverse u-sub"

Normal straightforward u-sub:

Calculate

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So

$$\int e^{\sqrt{x}}\ dx = \int e^u(2u)\ du = 2ue^u - 2\int e^u\ du$$

(integration by parts with $f(u) = 2u$ and $g'(u) = e^u$)

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

Trig substitution, or “reverse u-sub”

How about

$$\int \sqrt{1 - x^2}\; dx?$$

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$$\int \sqrt{1-x^2}\,dx?$$

We could try letting $u = 1 - x^2$, so that $x = \sqrt{1-u}$.

Trig substitution, or "reverse u-sub"

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$$\int \sqrt{1-x^2}\ dx?$$

We could try letting $u = 1 - x^2$, so that $x = \sqrt{1-u}$. Further, $du = -2x\ dx = -2\sqrt{1-u}\ du$. So

$$\int \sqrt{1-x^2}\ dx = -\tfrac{1}{2}\int \frac{\sqrt{u}}{\sqrt{1-u}}\ du = -\tfrac{1}{2}\int \sqrt{\frac{u}{1-u}}\ du \ldots$$

Trig substitution, or "reverse u-sub"

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$$\int \sqrt{1 - x^2}\, dx?$$

Trig substitution, or "reverse u-sub"

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Instead of using a substitution that looks like $u = f(x)$, we can try making a substitution that looks like $x = f(u)$.

Trig substitution, or “reverse u-sub”

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$$\int \sqrt{1 - x^2}\, dx?$$

Instead of using a substitution that looks like $u = f(x)$, we can try making a substitution that looks like $x = f(u)$. Is there a function $f(u)$ such that $1 - f^2(u)$ is a perfect square?

Trig substitution, or “reverse u-sub”

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Instead of using a substitution that looks like $u = f(x)$, we can try making a substitution that looks like $x = f(u)$. Is there a function $f(u)$ such that $1 - f^2(u)$ is a perfect square? Think

$$\cos^2(u) + \sin^2(u) = 1.$$

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Let $x = \cos(u)$.

Trig substitution, or "reverse u-sub"

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$$\cos^2(u) + \sin^2(u) = 1.$$

Let $x = \cos(u)$. Then $dx = -\sin(u)\, du$.

Trig substitution, or “reverse u-sub”

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$$\cos^2(u) + \sin^2(u) = 1.$$

Let $x = \cos(u)$. Then $dx = -\sin(u)\, du$. So

$$\int \sqrt{1 - x^2}\, dx = \int \sqrt{1 - \cos^2(u)} \cdot (-\sin(u))\, du$$

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$$= -\int \sqrt{\sin^2(u)} \cdot \sin(u)\, du = -\int \sin^2(u)\, du$$

$$= \tfrac{1}{2} \int \cos(2u) - 1\, du$$

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Let $x = \cos(u)$. Then $dx = -\sin(u)\, du$. So

$$\begin{aligned}\int \sqrt{1 - x^2}\, dx &= \int \sqrt{1 - \cos^2(u)} \cdot (-\sin(u))\, du \\ &= -\int \sqrt{\sin^2(u)} \cdot \sin(u)\, du = -\int \sin^2(u)\, du \\ &= \tfrac{1}{2} \int \cos(2u) - 1\, du = \tfrac{1}{2}(\tfrac{1}{2}\sin(2u) - u) + C\end{aligned}$$

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$$\int \sqrt{1 - x^2}\ dx = \int \sqrt{1 - \cos^2(u)} \cdot (-\sin(u))\ du$$

$$= -\int \sqrt{\sin^2(u)} \cdot \sin(u)\ du = -\int \sin^2(u)\ du$$

$$= \frac{1}{2} \int \cos(2u) - 1\ du = \frac{1}{2}(\frac{1}{2} \sin(2u) - u) + C$$

$$= \frac{1}{4} \sin(2\cos^{-1}(x))) - \frac{1}{2} \cos^{-1}(x) + C.$$

Simplifying $\sin(2\sin^{-1}(x)))$

So we don't get too bogged down, let's go back to writing this as

$$\sin(2u), \text{ where } \cos(u) = x \text{ so that } u = \cos^{-1}(x).$$

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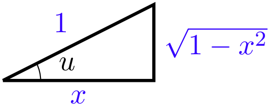

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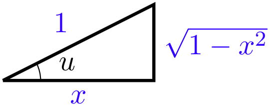

$$\sin(2u) = 2\sin(u)\cos(u) = 2\sqrt{1-x^2}\cdot x.$$

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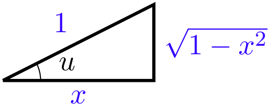

$$\sin(2u) = 2\sin(u)\cos(u) = 2\sqrt{1-x^2}\cdot x.$$

So

$$\int \sqrt{1-x^2}\,dx = \cdots = -\tfrac{1}{2}\cos^{-1}(x) + \tfrac{1}{4}\sin(2\sin^{-1}(x))) + C$$

$$= \tfrac{1}{2}x\sqrt{1-x^2} - \tfrac{1}{2}\cos^{-1}(x) + C$$

Check against geometry:

$$\boxed{\int \sqrt{1-x^2}\;dx = \frac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right) + C}$$

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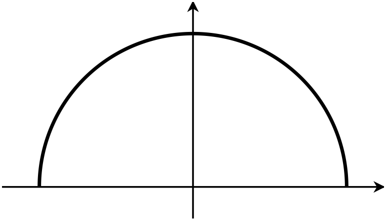
$$\boxed{\int \sqrt{1-x^2}\ dx = \frac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right) + C}$$

Note: $y = \sqrt{1-x^2}$ is the upper half of the graph of $y^2 + x^2 = 1$:



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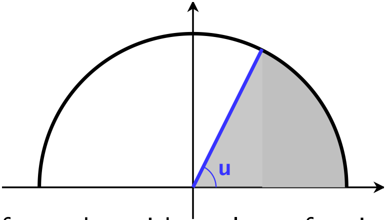

Recall: the area of a wedge with angle u of a circle or radius r is

$$A = (u/2\pi)\pi r^2 = \frac{1}{2}ur^2.$$

Check against geometry:

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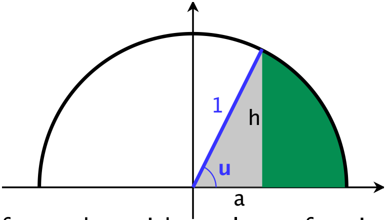

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So, for example, the integral $I = \int_a^1 \sqrt{1-x^2}\;dx$ should be

$$\text{(area of the wedge)} - \text{(area of the triangle)} = \tfrac{1}{2}u - \tfrac{1}{2}ah.$$

Check against geometry:

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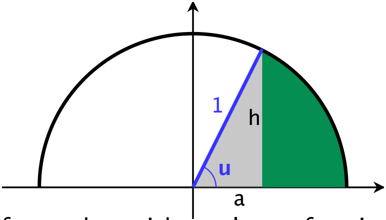

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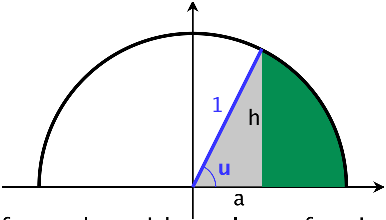

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Since $h = \sqrt{1-a^2}$ and $u = \cos^{-1}(a)$, we have

$$I = \tfrac{1}{2}\cos^{-1}(a) - \tfrac{1}{2}a\sqrt{1-a^2}.$$

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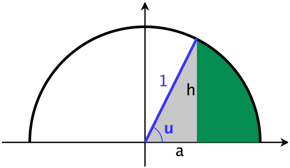
$$\boxed{\int \sqrt{1-x^2}\ dx = \frac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right) + C}$$



Geometrically,

$$\int_a^1 \sqrt{1-x^2}\ dx = \tfrac{1}{2}u - \tfrac{1}{2}ah = \tfrac{1}{2}\cos^{-1}(a) - \tfrac{1}{2}a\sqrt{1-a^2}.$$

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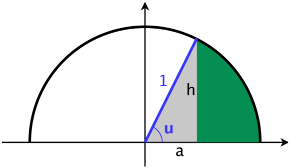

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Checking against the formula we computed:

$$\int_a^1 \sqrt{1-x^2}\ dx = \frac{1}{2}\left(x\sqrt{1-x^2} - \cos^{-1}(x)\right)\Big|_{x=a}^{1}$$

Check against geometry:

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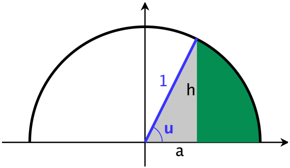

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$$= \frac{1}{2}\left((1\cdot 0 - \cos^{-1}(1)) - (a\sqrt{1-a^2} - \cos^{-1}(a))\right)$$

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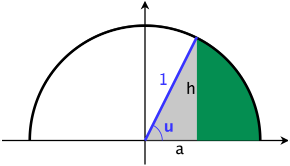

Geometrically,

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$$= \frac{1}{2}\cos^{-1}(a) - \frac{1}{2}a\sqrt{1-a^2} \quad \checkmark \quad (\cos^{-1}(1) = 0).$$

You try:

Calculate the following integrals using the suggested substitution. Be sure to simplify your answers.

1. $\int \frac{\sqrt{1-x^2}}{x^2}\ dx$ using $x = \sin(u)$

2. $\int \frac{1}{x^2\sqrt{1+x^2}}\ dx$ using $x = \tan(u)$

3. $\int \frac{x}{\sqrt{1+x^2}}\ dx$ two ways:

   (a) Let $x = \tan(u)$ (b) Let $u = 1 + x^2$.

You try:

Calculate the following integrals using the suggested substitution. Be sure to simplify your answers.

1. $\int \frac{\sqrt{1-x^2}}{x^2}\, dx$ using $x = \sin(u)$

$$= -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}(x) + C$$

2. $\int \frac{1}{x^2\sqrt{1+x^2}}\, dx$ using $x = \tan(u)$

$$= -\frac{\sqrt{1+x^2}}{x} + C.$$

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$$= \sqrt{1+x^2} + C$$

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Calculate the following integrals using the suggested substitution. Be sure to simplify your answers.

1. $\int \frac{\sqrt{1-x^2}}{x^2}\ dx$ using $x = \sin(u)$: $dx = \cos(u)\ du$, so
   $I = \int \frac{\sqrt{1-\sin^2(u)}}{\sin^2(u)} \cdot \cos(u)\ du = \int \frac{\cos^2(x)}{\sin^2(u)}\ du$
   $= \int \cot^2(u)\ du = -\cot(u) - u + C$
   $= -\cot(\sin^{-1}(x)) - \sin^{-1}(x) + C = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}(x) + C$
2. $\int \frac{1}{x^2\sqrt{1+x^2}}\ dx$ using $x = \tan(u)$: $dx = \sec^2(u)\ du$, so
   $I = \int \frac{1}{\tan^2(u)\sqrt{1+\tan^2(u)}}\ \sec^2(u)\ du = \int \frac{\sec^2(u)}{\tan^2(u)\sec(u)}\ du$
   $= \int \frac{\cos(u)}{\sin^2(u)}\ du = -\sin^{-1}(u) + C = -\csc(\tan^{-1}(x)) + C$
   $= -\frac{\sqrt{1+x^2}}{x} + C.$
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   (a) Let $x = \tan(u)$: $\quad dx = \sec^2(u)\ du$, so that

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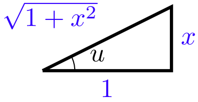

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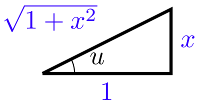


   (b) Let $u = 1 + x^2$: Then $du = 2x\, dx$, so that

$$\int \frac{x}{\sqrt{1+x^2}}\, dx = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1+x^2} + C.$$

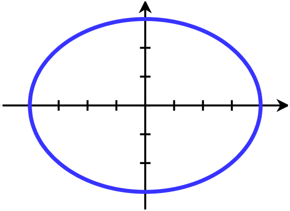
Another geometric example

Compute the area of an ellipse with minor radius 3 and major radius 4.

Another geometric example

Compute the area of an ellipse with minor radius 3 and major radius 4.

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$



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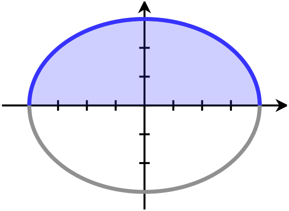

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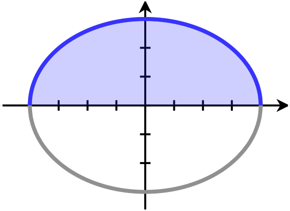

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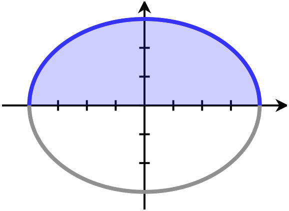

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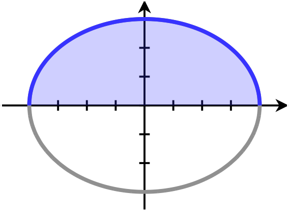

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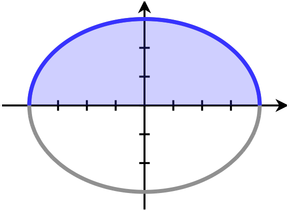

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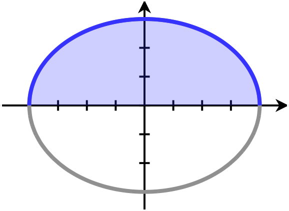

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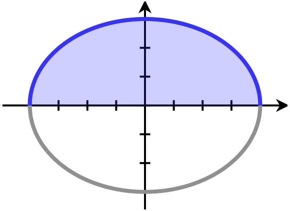

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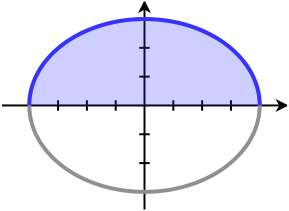

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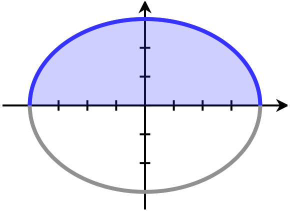

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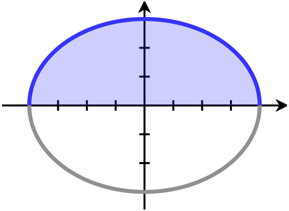

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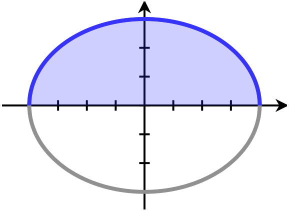

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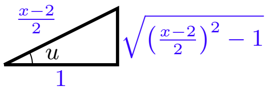

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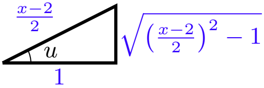

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Calculate the following integrals. Be sure to simplify your answers. Remember, the Pythagorean identities are

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