Today: 6.2 Trig integrals

Recall the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Warm up: Use the identities above to do the following problems.

1. Rewrite the following in terms of $\cos(x)$ and $\sin(x)$.

   (a) $\cos(2x)$ (b) $\sin(2x)$ (c) $\cos(x/2)$ (d) $\sin(x/2)$

2. (prep for u-sub) Rewrite the following as sums of terms that either have one $\cos(x)$ or one $\sin(x)$ each. For example,

   $$\cos^3(x) = (1 - \sin^2(x))\cos(x) = \boxed{\cos(x) - \sin^2(x)\cos(x)}.$$

   (a) $\sin^3(x)\cos^2(x)$ (b) $\sin^4(x)\cos^7(x)$ (c) $\cos^2(x/2)$

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

so that $\cos(x/2) = \sqrt{\frac{1}{2}(1+\cos(x))}$. Similarly,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= (1 - \sin^2(x/2)) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

so that $\sin(x/2) = \sqrt{\frac{1}{2}(1-\cos(x))}$.

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$.
Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

and since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2 - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

we have

$$\boxed{\cos^2(x) = \tfrac{1}{2}(1+\cos(2x))}.$$

Similarly, since

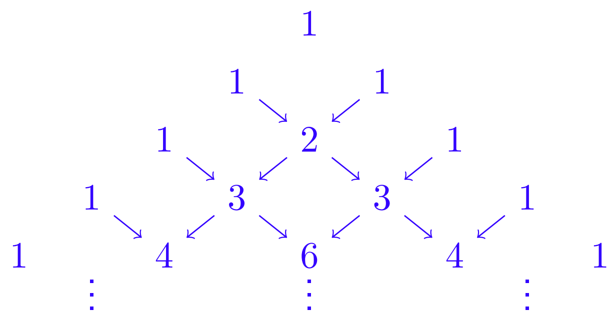
$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x),$$

we have

$$\boxed{\sin^2(x) = \tfrac{1}{2}(1-\cos(2x))}.$$

Reviewing fast polynomial expansion

Pascal's triangle is built recursively.
Start with a 1 in the 0th row.
The ith row starts and ends with 1, has $i+1$ numbers in it.
To get the other entries, add the jth and $j+1$st entry of the previous row to get the jth entry of the current row.

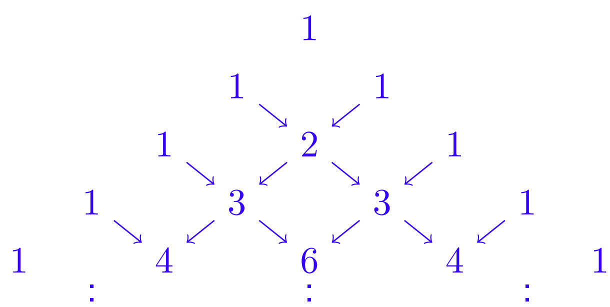


Then

$$(a+b)^k = c_0 a^k + c_1 a^{k-1} b + c_2 a^{k-2} b^2 + \cdots + c_{k-1} a b^{k-1} + c_k b^k$$

where c_j is the jth entry in the kth row. (Count from 0)

Reviewing fast polynomial expansion



Then

$$(a+b)^k = c_0 a^k + c_1 a^{k-1} b + c_2 a^{k-2} b^2 + \cdots + c_{k-1} a b^{k-1} + c_k b^k$$

where c_j is the jth entry in the kth row. (Count from 0)
For example,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(1-x)^4 = 1^4 + 4 \cdot 1^3 \cdot (-x) + 6 \cdot 1^2 \cdot (-x)^2 + 4 \cdot 1 \cdot (-x)^3 + (-x)^4$$
$$= 1 - 4x + 6x^2 - 4x^3 + x^4$$

Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

Computing $\int \sin^2(x)\cos(x)\ dx$:

Let $u = \sin(x)$, so $du = \cos(x)\ dx$.

Thus

$$\int \sin^2(x)\cos(x)\ dx = \int u^2\ du = \tfrac{1}{3}u^3 + C = -\tfrac{1}{3}\sin^3(x) + C.$$

Therefore

$$\int \cos^3(x)\ dx = \sin(x) - \tfrac{1}{3}\sin^3(x) + D.$$

You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx$

2. $\int \sin^4(x)\cos^7(x)\ dx$

3. $\int \cos^2(x)\ dx$

   (Hint for (3): replace $x/2$ with x from the warmup)

You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx \ = \int \sin(x)(1-\cos^2(x))\cos^2(x)\ dx =$
   $\int \sin(x)\cos^2(x) - \sin(x)\cos^4(x)\ dx = -\frac{1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$
2. $\int \sin^4(x)\cos^7(x)\ dx \ = \int \sin^4(x)(1-\sin^2(x))^3\cos(x)\ dx.$
   Let $u = \sin(x)$, so $du = \cos(x)\ dx$. So
   $I = \int u^4(1-u^2)^3\ du = \int u^4 - 3u^6 + 3u^8 - u^{10}\ du$
   $= \frac{1}{5}\sin^5(x) - \frac{3}{7}\sin^7(x) + \frac{3}{9}\sin^9(x) - \frac{1}{11}\sin^{11}(x) + C.$
3. $\int \cos^2(x)\ dx = \frac{1}{2}\int 1 + \cos(2x)\ dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$
   (Hint for (3): replace $x/2$ with x from the warmup)

Note that the $\cos^3(x)$ example from before, and 1. and 2. above, all have an odd number of one trig function and an even number of the other. But 3. has just has an even number of each.

Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)).$$

$$\int \sin^4(x)\ dx = \int \left(\tfrac{1}{2}(1-\cos(2x))\right)^2\ dx$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \underbrace{\frac{1}{4}\int \cos^2(2x)\ dx}_{(*)\text{ do it again!}}$$

$(*)$ $\frac{1}{4}\int \cos^2(2x)\ dx = \frac{1}{4}\cdot\frac{1}{2}\int 1+\cos(4x)\ dx = \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin(4x) + C$

So $\int \sin^4(x)\ dx = \boxed{\frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin(4x) + C}.$

In general, to solve $\int \cos^a(x)\sin^b(x)\ dx$:

1. If one of a or b is odd:
   (a) if a is odd: then $a-1$ is even, so rewrite
   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2}\cos(x) = (1-\sin^2(x))^{(a-1)/2}\cos(x)$$
   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.
   (b) if a is even, but b is odd: then, similarly, $b-1$ is even, so rewrite
   $$\sin^b(x) = (\sin^2(x))^{(b-1)/2}\sin(x) = (1-\cos^2(x))^{(b-1)/2}\sin(x)$$
   converting all but one of the $\sin(x)$'s into $\cos(x)$'s.
2. If both a and b are even: use the formula
   $$\cos^2(x) + \sin^2(x) = 1$$
   to turn all the $\cos(x)$'s into $\sin(x)$'s or vice versa, and then
   $$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \quad \text{or} \quad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$$
   to reduce the degrees of the terms down until you can go back to case 1.

You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx$
2. $\int \cos^3(x)\sin^3(x)\ dx$
3. $\int \cos^2(x)\sin^2(x)\ dx$
4. $\int \cos(x)\sin^5(x)\ dx$

You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx\ = \int (\frac{1}{2}(1+\cos(2x)))^2\ dx$
$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$$
$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

2. $\int \cos^3(x)\sin^3(x)\ dx = \int \cos^3(x)(1-\cos^2(x))\sin(x)\ dx$
$$= -\frac{1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C,$$
or $\int \cos^3(x)\sin^3(x)\ dx = \int \sin^3(x)(1-\sin^2(x))\cos(x)\ dx$
$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

3. $\int \cos^2(x)\sin^2(x)\ dx = \int (\cos(x)\sin(x))^2\ dx$
$$= \int (\frac{1}{2}\sin(2x))^2\ dx = \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\ dx$$
$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

4. $\int \cos(x)\sin^5(x)\ dx$
$$= \frac{1}{6}\sin^6(x) + C$$

Integrals with $\tan(x)$ and $\sec(x)$

Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.
Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$
$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}.$$

So let

$u = \sec(x) + \tan(x)$ so that $du = (\sec(x)\tan(x) + \sec^2(x))\ dx$.

So $$\int \sec(x)\ dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\ dx = \int u^{-1}\ du$$
$$= \ln|u| + C = \boxed{\ln|\sec(x)+\tan(x)| + C}.$$

Integrals with $\tan(x)$ and $\sec(x)$

Dividing both sites of

$$\sin^2(x) + \cos^2(x) = 1$$

by $\cos^2(x)$ gives

$$\tan^2(x) + 1 = \sec^2(x).$$

The more subtle difference with integrals with $\tan(x)$ and $\sec(x)$ is the slightly more complicated derivatives:

$$\frac{d}{dx}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

The guidelines therefore become less straightforward as well...

Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

In this case there isn't an immediate u-substitution to make. We might be able to use some of our new tools, but let's back up...

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

Let $u = \tan(x)$ so that $du = \sec^2(x)\ dx$. Then

$$I = \int u^4\ du + \int u^2\ du = \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5(x) + \tfrac{1}{3}\tan^3(x) + C.$$

Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

letting $u = \tan(x)$, so that $du = \sec^2(x)\ dx$;

or letting $u = \sec(x)$, so that $du = \sec(x)\tan(x)\ dx$.

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx = \int u^4 - u^2 du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5(x) - \tfrac{1}{3}\sec^3(x) + C$$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Answers: (1) $\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C$, (2) $\frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C$, ...

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Guidelines:

a. If I have an even number of $\sec(x)$'s, I want to turn all but 2 of them into $\tan(x)$'s and use $u = \tan(x)$.

b. If I have an odd number of $\tan(x)$'s, I want to turn all but 1 of them into $\sec(x)$'s and
   i. use $u = \sec(x)$ for terms that look like $\tan(x)\sec^x(a)$ (for $a > 0$), and
   ii. integrate any solitary $\tan(x)$ term by letting $u = \cos(x)$.

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1. $\int \tan^3(x)\sec^5(x)\ dx = \int \sec^5(x)(\sec^2(x) - 1)\tan(x)\ dx$
   $= \int (\sec^6(x) - \sec^4(x))\sec(x)\tan(x)\ dx$
   $= \cdots = \boxed{\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C}.$

2. $\int \tan^4(x)\sec^4(x)\ dx = \int \tan^4(x)(\tan^2(x) + 1)\sec^2(x)\ dx$
   $= \int (\tan^6(x) + \tan^4(x))\sec^2(x)\ dx$
   $= \cdots = \boxed{\frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C}.$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\ dx$$

$$= \int (\tan^5(x)+\tan^3(x))\sec^2(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x)+\tfrac{1}{4}\tan^4(x)+C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan(x)(\sec^2(x)-1)\sec^4(x)\ dx$$

$$= \int (\sec^5(x)-\sec^3(x))\sec(x)\tan(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x)-\tfrac{1}{4}\sec^4(x)+C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x)+3\tan^4(x)+3\tan^2(x)+1$$

and $\quad \sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x)+2\tan^2(x)+1$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x)+\tfrac{1}{4}\tan^4(x)+C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x)-\tfrac{1}{4}\sec^4(x)+C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x)+3\tan^4(x)+3\tan^2(x)+1$$

and $\quad \sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x)+2\tan^2(x)+1$

So

$$\begin{aligned}\tfrac{1}{6}\sec^6(x)-\tfrac{1}{4}\sec^4(x) &= \tfrac{1}{6}(\tan^6(x)+3\tan^4(x)+3\tan^2(x)+1) \\ &\quad -\tfrac{1}{4}(\tan^4(x)+2\tan^2(x)+1) \\ &= \tfrac{1}{6}\tan^6(x)+(\tfrac{1}{2}-\tfrac{1}{4})\tan^4(x)+(\tfrac{1}{2}-\tfrac{1}{2})\tan^2(x)+(\tfrac{1}{6}-\tfrac{1}{4}) \\ &= \tfrac{1}{6}\tan^6(x)+\tfrac{1}{4}\tan^4(x)+\text{const.} \quad \checkmark\end{aligned}$$

Example: $I = \int \sec^3(x)\ dx$

Uh oh... I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$\begin{aligned} I = \int \sec(x) \cdot \sec^2(x)\ dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\ dx \\ &= \sec(x)\tan(x) - I + \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

Thus $2I = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$, giving

$$\int \sec^3(x)\ dx = \boxed{\tfrac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + D}. \quad (D = C/2)$$

You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx$

2. $\int \tan^3(x)\ dx$

3. $\int \tan^2(x) \sec(x)\ dx$

4. $\int \sec^2(x) \tan(x)\ dx$

5. $\int \cot^3(x)\ dx$

You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$

   $= \tan(x) - x + C.$

2. $\int \tan^3(x)\ dx = \int (\sec^2(x) - 1) \tan(x)\ dx$

   $= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$

3. $\int \tan^2(x)\sec(x)\ dx = \int (\sec^2(x) - 1)\sec(x)\ dx$

   $= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$

4. $\int \sec^2(x)\tan(x)\ dx$ Let $u = \tan(x)$, so $du = \sec^2(x)$:

   $= \frac{1}{2}\tan^2(x) + C.$

5. $\int \cot^3(x)\ dx$ Using $1 + \cot^2(x) = \csc^2(x)$,

   $\int \cot^3(x)\ dx = \int (\csc^2(x) - 1)\cot(x)\ dx$

   $= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$