# Today: 6.2 Trig integrals

Recall the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Warm up: Use the identities above to do the following problems.

1. Rewrite the following in terms of $\cos(x)$ and $\sin(x)$.

   (a) $\cos(2x)$ (b) $\sin(2x)$ (c) $\cos(x/2)$ (d) $\sin(x/2)$

2. (prep for $u$-sub) Rewrite the following as sums of terms that either have one $\cos(x)$ or one $\sin(x)$ each. For example,

$$\cos^3(x) = (1 - \sin^2(x))\cos(x) = \boxed{\cos(x) - \sin^2(x)\cos(x)}.$$

   (a) $\sin^3(x)\cos^2(x)$ (b) $\sin^4(x)\cos^7(x)$ (c) $\cos^2(x/2)$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$
$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$
$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

so that $\cos(x/2) = \sqrt{\frac{1}{2}(1 + \cos(x))}$.

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

so that $\cos(x/2) = \sqrt{\frac{1}{2}(1 + \cos(x))}$. Similarly,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

so that $\cos(x/2) = \sqrt{\frac{1}{2}(1+\cos(x))}$. Similarly,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= (1 - \sin^2(x/2)) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\cos(x)\sin(x).$$

Further,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= \cos^2(x/2) - (1 - \cos^2(x/2)) = 2\cos^2(x/2) - 1$$

so that $\cos(x/2) = \sqrt{\frac{1}{2}(1 + \cos(x))}$. Similarly,

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2)$$

$$= (1 - \sin^2(x/2)) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

so that $\sin(x/2) = \sqrt{\frac{1}{2}(1 - \cos(x))}$.

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$.

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$. Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$.
Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

and since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2 - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$.
Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

and since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2 - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

we have

$$\boxed{\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x))}.$$

Using the identities

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$. Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

and since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2 - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

we have

$$\boxed{\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x))}.$$

Similarly, since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x),$$

Using the identities
$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin^2(x) + \cos^2(x) = 1,$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

$$\text{and} \quad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we have $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\sin(2x) = 2\cos(x)\sin(x)$.
Actually, the identities we'll use more for integration are

$$\boxed{\cos(x)\sin(x) = \tfrac{1}{2}\sin(2x)},$$

and since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2 - (1 - \cos^2(x)) = 2\cos^2(x) - 1,$$

we have

$$\boxed{\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x))}.$$

Similarly, since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x),$$

we have

$$\boxed{\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x))}.$$

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

$$1$$

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i + 1$ numbers in it.

$$
\begin{array}{ccc}
 & 1 & \\
1 & & 1
\end{array}
$$

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i + 1$ numbers in it.

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & & & 1
\end{array}
$$

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i+1$ numbers in it.

To get the other entries, add the $j$th and $j+1$st entry of the previous row to get the $j$th entry of the current row.

$$1$$

$$1 \qquad 1$$

$$1 \qquad\qquad 1$$

# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i + 1$ numbers in it.

To get the other entries, add the $j$th and $j + 1$st entry of the previous row to get the $j$th entry of the current row.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

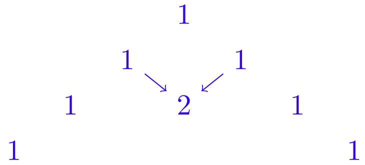
# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i + 1$ numbers in it.

To get the other entries, add the $j$th and $j + 1$st entry of the previous row to get the $j$th entry of the current row.



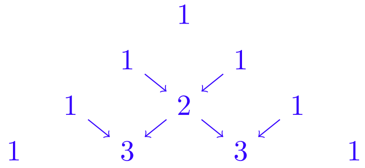
# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

The $i$th row starts and ends with $1$, has $i+1$ numbers in it.

To get the other entries, add the $j$th and $j+1$st entry of the previous row to get the $j$th entry of the current row.



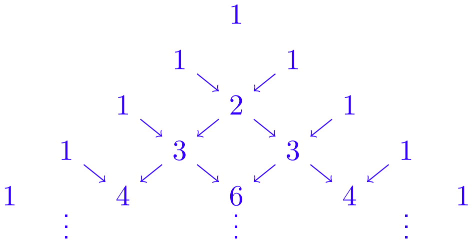
# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.

Start with a $1$ in the $0$th row.

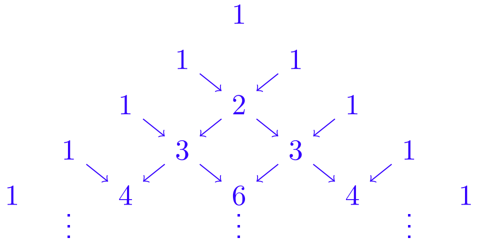
The $i$th row starts and ends with $1$, has $i+1$ numbers in it.

To get the other entries, add the $j$th and $j+1$st entry of the previous row to get the $j$th entry of the current row.



# Reviewing fast polynomial expansion

Pascal's triangle is built recursively.
Start with a $1$ in the $0$th row.
The $i$th row starts and ends with $1$, has $i+1$ numbers in it.
To get the other entries, add the $j$th and $j+1$st entry of the previous row to get the $j$th entry of the current row.

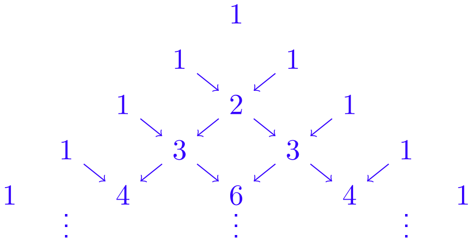



Then

$$(a+b)^k = c_0 a^k + c_1 a^{k-1} b + c_2 a^{k-2} b^2 + \cdots + c_{k-1} a b^{k-1} + c_k b^k$$

where $c_j$ is the $j$th entry in the $k$th row. (Count from 0)

# Reviewing fast polynomial expansion



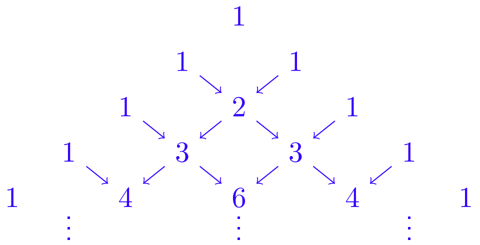

Then

$$(a+b)^k = c_0 a^k + c_1 a^{k-1} b + c_2 a^{k-2} b^2 + \cdots + c_{k-1} a b^{k-1} + c_k b^k$$

where $c_j$ is the $j$th entry in the $k$th row. (Count from 0)
For example,

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

# Reviewing fast polynomial expansion




Then

$$(a+b)^k = c_0 a^k + c_1 a^{k-1} b + c_2 a^{k-2} b^2 + \cdots + c_{k-1} a b^{k-1} + c_k b^k$$

where $c_j$ is the $j$th entry in the $k$th row. (Count from 0)
For example,

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$\begin{aligned}(1-x)^4 &= 1^4 + 4 \cdot 1^3 \cdot (-x) + 6 \cdot 1^2 \cdot (-x)^2 + 4 \cdot 1 \cdot (-x)^3 + (-x)^4 \\ &= 1 - 4x + 6x^2 - 4x^3 + x^4\end{aligned}$$

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

Computing $\int \sin^2(x)\cos(x)\ dx$ :

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

Computing $\int \sin^2(x)\cos(x)\ dx$ :

Let $u = \sin(x)$, so $du = \cos(x)\ dx$.

Thus

$$\int \sin^2(x)\cos(x)\ dx = \int u^2\ du$$

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

Computing $\int \sin^2(x)\cos(x)\ dx$ :

$$\text{Let } u = \sin(x), \qquad \text{so } du = \cos(x)\ dx.$$

Thus

$$\int \sin^2(x)\cos(x)\ dx = \int u^2\ du = \tfrac{1}{3}u^3 + C = -\tfrac{1}{3}\sin^3(x) + C.$$

# Using trigonometric identities for integration

We saw that

$$\cos^3(x) = \cos(x) - \sin^2(x)\cos(x).$$

Thus

$$\int \cos^3(x)\ dx = \int \cos(x)\ dx - \int \sin^2(x)\cos(x)\ dx.$$

Computing $\int \sin^2(x)\cos(x)\ dx$ :

Let $u = \sin(x)$, so $du = \cos(x)\ dx$.

Thus

$$\int \sin^2(x)\cos(x)\ dx = \int u^2\ du = \tfrac{1}{3}u^3 + C = -\tfrac{1}{3}\sin^3(x) + C.$$

Therefore

$$\int \cos^3(x)\ dx = \sin(x) - \tfrac{1}{3}\sin^3(x) + D.$$

# You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx$

2. $\int \sin^4(x)\cos^7(x)\ dx$

3. $\int \cos^2(x)\ dx$

   (Hint for (3): replace $x/2$ with $x$ from the warmup)

# You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx$

   $= -\frac{1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$

2. $\int \sin^4(x)\cos^7(x)\ dx$

   $= \frac{1}{5}\sin^5(x) - \frac{3}{7}\sin^7(x) + \frac{3}{9}\sin^9(x) - \frac{1}{11}\sin^{11}(x) + C.$

3. $\int \cos^2(x)\ dx$ $= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$

   (Hint for (3): replace $x/2$ with $x$ from the warmup)

# You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx\ = \int \sin(x)(1-\cos^2(x))\cos^2(x)\ dx =$
   $\int \sin(x)\cos^2(x) - \sin(x)\cos^4(x)\ dx = -\frac{1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$
2. $\int \sin^4(x)\cos^7(x)\ dx\ = \int \sin^4(x)(1-\sin^2(x))^3\cos(x)\ dx.$
   Let $u = \sin(x)$, so $du = \cos(x)\ dx$. So
   $I = \int u^4(1-u^2)^3\ du = \int u^4 - 3u^6 + 3u^8 - u^{10}\ du$
   $= \frac{1}{5}\sin^5(x) - \frac{3}{7}\sin^7(x) + \frac{3}{9}\sin^9(x) - \frac{1}{11}\sin^{11}(x) + C.$
3. $\int \cos^2(x)\ dx = \frac{1}{2}\int 1 + \cos(2x)\ dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$
   (Hint for (3): replace $x/2$ with $x$ from the warmup)

# You try

Compute the following antiderivatives.
You'll want to use the warmup for all three.

1. $\int \sin^3(x)\cos^2(x)\ dx\ = \int \sin(x)(1-\cos^2(x))\cos^2(x)\ dx =$
   $\int \sin(x)\cos^2(x) - \sin(x)\cos^4(x)\ dx = -\frac{1}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C.$
2. $\int \sin^4(x)\cos^7(x)\ dx\ = \int \sin^4(x)(1-\sin^2(x))^3\cos(x)\ dx.$
   Let $u = \sin(x)$, so $du = \cos(x)\ dx$. So
   $I = \int u^4(1-u^2)^3\ du = \int u^4 - 3u^6 + 3u^8 - u^{10}\ du$
   $= \frac{1}{5}\sin^5(x) - \frac{3}{7}\sin^7(x) + \frac{3}{9}\sin^9(x) - \frac{1}{11}\sin^{11}(x) + C.$
3. $\int \cos^2(x)\ dx = \frac{1}{2}\int 1 + \cos(2x)\ dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$
   (Hint for (3): replace $x/2$ with $x$ from the warmup)

Note that the $\cos^3(x)$ example from before, and 1. and 2. above, all have an odd number of one trig function and an even number of the other. But 3. has just has an even number of each.

Compute $\int \sin^4(x)\ dx$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)).$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1 - \cos(2x)))^2\ dx$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1 - \cos(2x)))^2\ dx$$

$$= \tfrac{1}{4} \int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1-\cos(2x)))^2\ dx$$

$$= \tfrac{1}{4} \int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \underbrace{\tfrac{1}{4}\int \cos^2(2x)\ dx}_{(*)\text{ do it again!}}$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula
$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1 - \cos(2x)))^2\ dx$$

$$= \tfrac{1}{4} \int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \underbrace{\tfrac{1}{4}\int \cos^2(2x)\ dx}_{(*)\ \text{do it again!}}$$

$(*)$ $$\tfrac{1}{4}\int \cos^2(2x)\, dx = \tfrac{1}{4}\cdot\tfrac{1}{2}\int 1+\cos(4x)\, dx$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1 - \cos(2x)))^2\ dx$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \underbrace{\frac{1}{4}\int \cos^2(2x)\ dx}_{(*)\text{ do it again!}}$$

$$(*)\ \frac{1}{4}\int \cos^2(2x)\, dx = \frac{1}{4}\cdot\frac{1}{2}\int 1+\cos(4x)\, dx = \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{4}\sin(4x) + C$$

# Compute $\int \sin^4(x)\ dx$

Here's another example where there's only an even number of $\sin(x)$'s/$\cos(x)$'s. So I'm going to use the half angle formula

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

$$\int \sin^4(x)\ dx = \int (\tfrac{1}{2}(1-\cos(2x)))^2\ dx$$

$$= \tfrac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\ dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \underbrace{\tfrac{1}{4}\int \cos^2(2x)\ dx}_{(*)\ \text{do it again!}}$$

$(*)$ $\tfrac{1}{4}\int \cos^2(2x)\,dx = \tfrac{1}{4}\cdot\tfrac{1}{2}\int 1+\cos(4x)\,dx = \tfrac{1}{8}x+\tfrac{1}{8}\cdot\tfrac{1}{4}\sin(4x)+C$

So $\int \sin^4(x)\ dx = \boxed{\tfrac{1}{4}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{8}x + \tfrac{1}{8}\cdot\tfrac{1}{4}\sin(4x) + C}.$

In general, to solve $\int \cos^a(x) \sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:

In general, to solve $\int \cos^a(x) \sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd:

In general, to solve $\int \cos^a(x) \sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite

   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2} \cos(x) = (1-\sin^2(x))^{(a-1)/2} \cos(x)$$

   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.

In general, to solve $\int \cos^a(x)\sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite

   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2}\cos(x) = (1-\sin^2(x))^{(a-1)/2}\cos(x)$$

   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.
   (b) if $a$ is even, but $b$ is odd:

In general, to solve $\int \cos^a(x)\sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite

   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2}\cos(x) = (1-\sin^2(x))^{(a-1)/2}\cos(x)$$

   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.
   (b) if $a$ is even, but $b$ is odd: then, similarly, $b-1$ is even, so rewrite

   $$\sin^b(x) = (\sin^2(x))^{(b-1)/2}\sin(x) = (1-\cos^2(x))^{(b-1)/2}\sin(x)$$

   converting all but one of the $\sin(x)$'s into $\cos(x)$'s.

In general, to solve $\int \cos^a(x)\sin^b(x)\, dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite

   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2}\cos(x) = (1-\sin^2(x))^{(a-1)/2}\cos(x)$$

   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.

   (b) if $a$ is even, but $b$ is odd: then, similarly, $b-1$ is even, so rewrite

   $$\sin^b(x) = (\sin^2(x))^{(b-1)/2}\sin(x) = (1-\cos^2(x))^{(b-1)/2}\sin(x)$$

   converting all but one of the $\sin(x)$'s into $\cos(x)$'s.
2. If both $a$ and $b$ are even:

In general, to solve $\int \cos^a(x) \sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite

   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2} \cos(x) = (1-\sin^2(x))^{(a-1)/2} \cos(x)$$

   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.

   (b) if $a$ is even, but $b$ is odd: then, similarly, $b-1$ is even, so rewrite

   $$\sin^b(x) = (\sin^2(x))^{(b-1)/2} \sin(x) = (1-\cos^2(x))^{(b-1)/2} \sin(x)$$

   converting all but one of the $\sin(x)$'s into $\cos(x)$'s.
2. If both $a$ and $b$ are even: use the formula

   $$\cos^2(x) + \sin^2(x) = 1$$

   to turn all the $\cos(x)$'s into $\sin(x)$'s or vice versa

In general, to solve $\int \cos^a(x)\sin^b(x)\ dx$ :

1. If one of $a$ or $b$ is odd:
   (a) if $a$ is odd: then $a-1$ is even, so rewrite
   $$\cos^a(x) = (\cos^2(x))^{(a-1)/2}\cos(x) = (1-\sin^2(x))^{(a-1)/2}\cos(x)$$
   converting all but one of the $\cos(x)$'s into $\sin(x)$'s.
   (b) if $a$ is even, but $b$ is odd: then, similarly, $b-1$ is even, so rewrite
   $$\sin^b(x) = (\sin^2(x))^{(b-1)/2}\sin(x) = (1-\cos^2(x))^{(b-1)/2}\sin(x)$$
   converting all but one of the $\sin(x)$'s into $\cos(x)$'s.
2. If both $a$ and $b$ are even: use the formula
   $$\cos^2(x) + \sin^2(x) = 1$$
   to turn all the $\cos(x)$'s into $\sin(x)$'s or vice versa, and then
   $$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x)) \quad \text{or} \quad \cos^2(x) = \tfrac{1}{2}(1+\cos(2x))$$
   to reduce the degrees of the terms down until you can go back to case 1.

# You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx$
2. $\int \cos^3(x)\sin^3(x)\ dx$
3. $\int \cos^2(x)\sin^2(x)\ dx$
4. $\int \cos(x)\sin^5(x)\ dx$

# You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx$

$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

2. $\int \cos^3(x)\sin^3(x)\ dx$

$$= -\frac{1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C,$$

or

$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

3. $\int \cos^2(x)\sin^2(x)\ dx$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

4. $\int \cos(x)\sin^5(x)\ dx$

$$= \frac{1}{6}\sin^6(x) + C$$

# You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx$ $\displaystyle = \int (\tfrac{1}{2}(1+\cos(2x)))^2\ dx$
$$= \tfrac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\ dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin(2x) + \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos(4x))\ dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin(2x) + \tfrac{1}{8}x + \tfrac{1}{32}\sin(4x) + C$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C.$$
2. $\int \cos^3(x)\sin^3(x)\ dx$
$$= -\tfrac{1}{4}\cos^4(x) + \tfrac{1}{6}\cos^6(x) + C,$$
or
$$= \tfrac{1}{4}\sin^4(x) - \tfrac{1}{6}\sin^6(x) + C$$
3. $\int \cos^2(x)\sin^2(x)\ dx$
$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin(4x) + C$$
4. $\int \cos(x)\sin^5(x)\ dx$
$$= \tfrac{1}{6}\sin^6(x) + C$$

# You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx \ = \int (\frac{1}{2}(1+\cos(2x)))^2\ dx$
$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$$
$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$
2. $\int \cos^3(x)\sin^3(x)\ dx = \int \cos^3(x)(1-\cos^2(x))\sin(x)\ dx$
$$= -\frac{1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C,$$
   or
$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$
3. $\int \cos^2(x)\sin^2(x)\ dx$
$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$
4. $\int \cos(x)\sin^5(x)\ dx$
$$= \frac{1}{6}\sin^6(x) + C$$

## You try

Compute the following integrals:

1. $\int \cos^4(x)\ dx \ = \int (\frac{1}{2}(1+\cos(2x)))^2\ dx$
$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\ dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$$
$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$
2. $\int \cos^3(x)\sin^3(x)\ dx = \int \cos^3(x)(1-\cos^2(x))\sin(x)\ dx$
$$= -\frac{1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C,$$
or $\int \cos^3(x)\sin^3(x)\ dx = \int \sin^3(x)(1-\sin^2(x))\cos(x)\ dx$
$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$
3. $\int \cos^2(x)\sin^2(x)\ dx = \int (\cos(x)\sin(x))^2\ dx$
$$= \int (\frac{1}{2}\sin(2x))^2\ dx = \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\ dx$$
$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$
4. $\int \cos(x)\sin^5(x)\ dx$
$$= \frac{1}{6}\sin^6(x) + C$$

# Integrals with $\tan(x)$ and $\sec(x)$

# Integrals with $\tan(x)$ and $\sec(x)$

Integrating $\tan(x)$:

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$.

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

## Integrating $\sec(x)$:

# Integrals with $\tan(x)$ and $\sec(x)$

Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}$$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

## Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}.$$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

## Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} = \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}.$$

So let

$u = \sec(x) + \tan(x)$ so that $du = (\sec(x)\tan(x) + \sec^2(x))\ dx.$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

## Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} = \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}.$$

So let

$u = \sec(x) + \tan(x)$ so that $du = (\sec(x)\tan(x) + \sec^2(x))\ dx$.

So $$\int \sec(x)\ dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\ dx = \int u^{-1}\ du$$

# Integrals with $\tan(x)$ and $\sec(x)$

## Integrating $\tan(x)$:

Note $\tan(x) = \sin(x)/\cos(x)$.

Let $u = \cos(x)$, so that $du = -\sin(x)\ dx$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)}\ dx = -\int u^{-1}\ du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C = \boxed{\ln|\sec(x)| + C}.$$

## Integrating $\sec(x)$: Note

$$\sec(x) = \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}.$$

So let

$$u = \sec(x)+\tan(x) \quad \text{so that} \quad du = (\sec(x)\tan(x)+\sec^2(x))\ dx.$$

So $$\int \sec(x)\ dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\ dx = \int u^{-1}\ du$$

$$= \ln|u| + C = \boxed{\ln|\sec(x)+\tan(x)| + C}.$$

# Integrals with $\tan(x)$ and $\sec(x)$

Dividing both sites of

$$\sin^2(x) + \cos^2(x) = 1$$

by $\cos^2(x)$ gives

$$\tan^2(x) + 1 = \sec^2(x).$$

# Integrals with $\tan(x)$ and $\sec(x)$

Dividing both sites of

$$\sin^2(x) + \cos^2(x) = 1$$

by $\cos^2(x)$ gives

$$\tan^2(x) + 1 = \sec^2(x).$$

The more subtle difference with integrals with $\tan(x)$ and $\sec(x)$ is the slightly more complicated derivatives:

$$\frac{d}{dx}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

# Integrals with $\tan(x)$ and $\sec(x)$

Dividing both sites of

$$\sin^2(x) + \cos^2(x) = 1$$

by $\cos^2(x)$ gives

$$\tan^2(x) + 1 = \sec^2(x).$$

The more subtle difference with integrals with $\tan(x)$ and $\sec(x)$ is the slightly more complicated derivatives:

$$\frac{d}{dx}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

The guidelines therefore become less straightforward as well...

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

In this case there isn't an immediate $u$-substitution to make. We might be able to use some of our new tools, but let's back up...

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

In this case there isn't an immediate $u$-substitution to make. We might be able to use some of our new tools, but let's back up. . .

---

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways:

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

In this case there isn't an immediate $u$-substitution to make. We might be able to use some of our new tools, but let's back up...

---

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$.

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

If I don't immediately recognize a trig integral, the first thing I do is convert everything into $\sin(x)$'s and $\cos(x)$'s:

$$\int \tan^2(x)\sec^4(x)\ dx = \int \frac{\sin^2(x)}{\cos^6(x)}\ dx.$$

In this case there isn't an immediate $u$-substitution to make. We might be able to use some of our new tools, but let's back up...

---

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

Let $u = \tan(x)$, so that $du = \sec^2(x)\ dx$;

or let $u = \sec(x)$, so that $du = \sec(x)\tan(x)\ dx$.

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

Let $u = \tan(x)$, so that $du = \sec^2(x)\ dx$;

or let $u = \sec(x)$, so that $du = \sec(x)\tan(x)\ dx$.

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

Let $u = \tan(x)$ so that $du = \sec^2(x)\ dx$.

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

Let $u = \tan(x)$, so that $du = \sec^2(x)\ dx$;

or let $u = \sec(x)$, so that $du = \sec(x)\tan(x)\ dx$.

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

Let $u = \tan(x)$ so that $du = \sec^2(x)\ dx$. Then

$$I = \int u^4\, du + \int u^2\, du$$

# Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

Let $u = \tan(x)$ so that $du = \sec^2(x)\ dx$. Then

$$I = \int u^4\, du + \int u^2\, du = \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C$$

## Example: compute $\int \tan^2(x)\sec^4(x)\ dx$

Instead, I can use

$$\tan^2(x) + 1 = \sec^2(x)$$

in one of two ways: to turn $\sec^2(x)$ into $\tan^2(x) + 1$, or to turn $\tan^2(x)$ into $\sec^2(x) - 1$. There are also two substitutions to keep in mind:

$$\text{Let } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or let } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$I = \int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)(\tan^2(x) + 1)\sec^2(x)\ dx$$

$$= \int \tan^4(x)\sec^2(x)\ dx + \int \tan^2(x)\sec^2(x)\ dx$$

Let $u = \tan(x)$ so that $du = \sec^2(x)\ dx$. Then

$$I = \int u^4\, du + \int u^2\, du = \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5(x) + \tfrac{1}{3}\tan^3(x) + C.$$

Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

## Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

# Example: compute $\int \tan^3(x) \sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx$$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

letting $u = \tan(x)$, so that $du = \sec^2(x)\ dx$;

or letting $u = \sec(x)$, so that $du = \sec(x)\tan(x)\ dx$.

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx = \int u^4 - u^2 du$$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx = \int u^4 - u^2 du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5(x) - \tfrac{1}{3}\sec^3(x) + C$$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx = \int u^4 - u^2 du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5(x) - \tfrac{1}{3}\sec^3(x) + C$$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

# Example: compute $\int \tan^3(x)\sec^3(x)\ dx$

Again, I have $\tan^2(x) + 1 = \sec^2(x)$, and the substitutions

$$\text{letting } u = \tan(x), \text{ so that } du = \sec^2(x)\ dx;$$

$$\text{or letting } u = \sec(x), \text{ so that } du = \sec(x)\tan(x)\ dx.$$

$$\int \tan^3(x)\sec^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\sec^3(x)\ dx$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)\ dx = \int u^4 - u^2 du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5(x) - \tfrac{1}{3}\sec^3(x) + C$$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Answers: (1) $\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C$, (2) $\frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C$, . . .

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Guidelines:

a. If I have an even number of $\sec(x)$'s, I want to turn all but 2 of them into $\tan(x)$'s and use $u = \tan(x)$.
b. If I have an odd number of $\tan(x)$'s, I want to turn all but 1 of them into $\sec(x)$'s and
   i. use $u = \sec(x)$ for terms that look like $\tan(x)\sec^x(a)$ (for $a > 0$), and
   ii. integrate any solitary $\tan(x)$ term by letting $u = \cos(x)$.

---

1. $\int \tan^3(x)\sec^5(x)\ dx$

2. $\int \tan^4(x)\sec^4(x)\ dx$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Guidelines:

a. If I have an even number of $\sec(x)$'s, I want to turn all but 2 of them into $\tan(x)$'s and use $u = \tan(x)$.
b. If I have an odd number of $\tan(x)$'s, I want to turn all but 1 of them into $\sec(x)$'s and
   i. use $u = \sec(x)$ for terms that look like $\tan(x)\sec^x(a)$ (for $a > 0$), and
   ii. integrate any solitary $\tan(x)$ term by letting $u = \cos(x)$.

---

1. $\int \tan^3(x)\sec^5(x)\ dx$

$$= \boxed{\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C}.$$

2. $\int \tan^4(x)\sec^4(x)\ dx$

$$= \boxed{\frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C}.$$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Guidelines:

a. If I have an even number of $\sec(x)$'s, I want to turn all but 2 of them into $\tan(x)$'s and use $u = \tan(x)$.
b. If I have an odd number of $\tan(x)$'s, I want to turn all but 1 of them into $\sec(x)$'s and
   i. use $u = \sec(x)$ for terms that look like $\tan(x)\sec^x(a)$ (for $a > 0$), and
   ii. integrate any solitary $\tan(x)$ term by letting $u = \cos(x)$.

---

1. $\int \tan^3(x)\sec^5(x)\;dx = \int \sec^5(x)(\sec^2(x) - 1)\tan(x)\;dx$
   $= \int (\sec^6(x) - \sec^4(x))\sec(x)\tan(x)\;dx$
   $$= \cdots = \boxed{\tfrac{1}{7}\sec^7(x) - \tfrac{1}{5}\sec^5(x) + C}.$$

2. $\int \tan^4(x)\sec^4(x)\;dx$

$$= \boxed{\tfrac{1}{7}\tan^7(x) + \tfrac{1}{5}\tan^5(x) + C}.$$

You try: Compute the antiderivatives of the following functions:

(1) $\tan^3(x)\sec^5(x)$ (2) $\tan^4(x)\sec^4(x)$ (3) $\tan^3(x)\sec^4(x)$

Guidelines:

a. If I have an even number of $\sec(x)$'s, I want to turn all but 2 of them into $\tan(x)$'s and use $u = \tan(x)$.
b. If I have an odd number of $\tan(x)$'s, I want to turn all but 1 of them into $\sec(x)$'s and
   i. use $u = \sec(x)$ for terms that look like $\tan(x)\sec^x(a)$ (for $a > 0$), and
   ii. integrate any solitary $\tan(x)$ term by letting $u = \cos(x)$.

---

1. $\int \tan^3(x)\sec^5(x)\ dx = \int \sec^5(x)(\sec^2(x) - 1)\tan(x)\ dx$
   $= \int (\sec^6(x) - \sec^4(x))\sec(x)\tan(x)\ dx$
   $= \cdots = \boxed{\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C}.$
2. $\int \tan^4(x)\sec^4(x)\ dx = \int \tan^4(x)(\tan^2(x) + 1)\sec^2(x)\ dx$
   $= \int (\tan^6(x) + \tan^4(x))\sec^2(x)\ dx$
   $= \cdots = \boxed{\frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C}.$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

b. Odd number of $\tan(x)$'s:

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\ dx$$

$$= \int (\tan^5(x)+\tan^3(x))\sec^2(x)\ dx = \boxed{\frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\ dx$$

$$= \int (\tan^5(x)+\tan^3(x))\sec^2(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x) + \tfrac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan(x)(\sec^2(x)-1)\sec^4(x)\ dx$$

$$= \int (\sec^5(x)-\sec^3(x))\sec(x)\tan(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) + C}.$$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\ dx$$

$$= \int (\tan^5(x)+\tan^3(x))\sec^2(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x) + \tfrac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \int \tan(x)(\sec^2(x)-1)\sec^4(x)\ dx$$

$$= \int (\sec^5(x)-\sec^3(x))\sec(x)\tan(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) + C}.$$

Why did we get seemingly different answers??

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\;dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\;dx$$

$$= \int (\tan^5(x)+\tan^3(x))\sec^2(x)\;dx = \boxed{\tfrac{1}{6}\tan^6(x)+\tfrac{1}{4}\tan^4(x)+C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\;dx = \int \tan(x)(\sec^2(x)-1)\sec^4(x)\;dx$$

$$= \int (\sec^5(x)-\sec^3(x))\sec(x)\tan(x)\;dx = \boxed{\tfrac{1}{6}\sec^6(x)-\tfrac{1}{4}\sec^4(x)+C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x)+3\tan^4(x)+3\tan^2(x)+1$$

and $\quad \sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x)+2\tan^2(x)+1$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x) + \tfrac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) + C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1$$

and $\quad \sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x) + 2\tan^2(x) + 1$

So

$$\begin{aligned}\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) = {} & \tfrac{1}{6}(\tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1) \\ & - \tfrac{1}{4}(\tan^4(x) + 2\tan^2(x) + 1)\end{aligned}$$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\tan^6(x) + \tfrac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) + C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1$$

and $$\sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x) + 2\tan^2(x) + 1$$

So

$$\begin{aligned}\tfrac{1}{6}\sec^6(x) - \tfrac{1}{4}\sec^4(x) &= \tfrac{1}{6}(\tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1) \\ &\quad - \tfrac{1}{4}(\tan^4(x) + 2\tan^2(x) + 1) \\ &= \tfrac{1}{6}\tan^6(x) + (\tfrac{1}{2} - \tfrac{1}{4})\tan^4(x) + (\tfrac{1}{2} - \tfrac{1}{2})\tan^2(x) + (\tfrac{1}{6} - \tfrac{1}{4})\end{aligned}$$

I can do (3) $\tan^3(x)\sec^4(x)$ either way:

a. Even number of $\sec(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^4(x) + C}.$$

b. Odd number of $\tan(x)$'s:

$$\int \tan^3(x)\sec^4(x)\ dx = \boxed{\frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) + C}.$$

Why did we get seemingly different answers?? Actually,

$$\sec^6(x) = (\tan^2(x)+1)^3 = \tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1$$

and $$\sec^4(x) = (\tan^2(x)+1)^2 = \tan^4(x) + 2\tan^2(x) + 1$$

So

$$\begin{aligned}\frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) &= \frac{1}{6}(\tan^6(x) + 3\tan^4(x) + 3\tan^2(x) + 1) \\ &\quad - \frac{1}{4}(\tan^4(x) + 2\tan^2(x) + 1) \\ &= \frac{1}{6}\tan^6(x) + (\tfrac{1}{2} - \tfrac{1}{4})\tan^4(x) + (\tfrac{1}{2} - \tfrac{1}{2})\tan^2(x) + (\tfrac{1}{6} - \tfrac{1}{4}) \\ &= \frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^4(x) + \text{const.} \quad \checkmark\end{aligned}$$

Example: $I = \int \sec^3(x)\ dx$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh... I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s.

# Example: $I = \int \sec^3(x)\ dx$

Uh oh. . . I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

# Example: $I = \int \sec^3(x)\ dx$

Uh oh...I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh...I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

# Example: $I = \int \sec^3(x)\ dx$

Uh oh. . . I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$I = \int \sec(x) \cdot \sec^2(x)\ dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx$$

## Example: $I = \int \sec^3(x)\, dx$

Uh oh... I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$I = \int \sec(x) \cdot \sec^2(x)\, dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\, dx$$

$$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\, dx$$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh. . . I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$I = \int \sec(x) \cdot \sec^2(x)\ dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx$$

$$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx$$

$$= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\ dx$$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh...I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$\begin{aligned} I = \int \sec(x) \cdot \sec^2(x)\ dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\ dx \\ &= \sec(x)\tan(x) - I + \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh. . . I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$\begin{aligned} I = \int \sec(x) \cdot \sec^2(x)\ dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\ dx \\ &= \sec(x)\tan(x) - I + \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

Thus $2I = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$

## Example: $I = \int \sec^3(x)\ dx$

Uh oh. . . I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$\begin{aligned} I = \int \sec(x) \cdot \sec^2(x)\ dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\ dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\ dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\ dx \\ &= \sec(x)\tan(x) - I + \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

Thus $2I = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$, giving

$$\int \sec^3(x)\ dx = \boxed{\tfrac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + C/2}.$$

## Example: $I = \int \sec^3(x)\, dx$

Uh oh... I neither have an even number of $\sec(x)$'s nor an odd number of $\tan(x)$'s. Integration by parts:

$$\text{Let } f(x) = \sec(x) \quad \text{and} \quad g'(x) = \sec^2(x),$$

$$\text{so that } f'(x) = \sec(x)\tan(x) \quad \text{and} \quad g'(x) = \tan(x).$$

So

$$\begin{aligned} I = \int \sec(x) \cdot \sec^2(x)\, dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\, dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\, dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)\, dx \\ &= \sec(x)\tan(x) - I + \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

Thus $2I = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$, giving

$$\int \sec^3(x)\, dx = \boxed{\tfrac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + D}. \ (D = C/2)$$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx$
2. $\int \tan^3(x)\ dx$
3. $\int \tan^2(x) \sec(x)\ dx$
4. $\int \sec^2(x) \tan(x)\ dx$
5. $\int \cot^3(x)\ dx$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx$

$$= \tan(x) - x + C.$$

2. $\int \tan^3(x)\ dx$

$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$$

3. $\int \tan^2(x)\sec(x)\ dx$

$$= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$$

4. $\int \sec^2(x)\tan(x)\ dx$

$$= \frac{1}{2}\tan^2(x) + C.$$

5. $\int \cot^3(x)\ dx$

$$= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$

$$= \tan(x) - x + C.$$

2. $\int \tan^3(x)\ dx$

$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$$

3. $\int \tan^2(x)\sec(x)\ dx$

$$= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$$

4. $\int \sec^2(x)\tan(x)\ dx$

$$= \frac{1}{2}\tan^2(x) + C.$$

5. $\int \cot^3(x)\ dx$

$$= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$

   $= \tan(x) - x + C.$

2. $\int \tan^3(x)\ dx = \int (\sec^2(x) - 1) \tan(x)\ dx$

   $= \frac{1}{2} \tan^2(x) + \ln|\cos(x)| + C.$

3. $\int \tan^2(x) \sec(x)\ dx$

   $= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$

4. $\int \sec^2(x) \tan(x)\ dx$

   $= \frac{1}{2} \tan^2(x) + C.$

5. $\int \cot^3(x)\ dx$

   $= -\frac{1}{2} \cot^2(x) - \ln|\sin(x)| + C.$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$
   $$= \tan(x) - x + C.$$
2. $\int \tan^3(x)\ dx = \int (\sec^2(x) - 1)\tan(x)\ dx$
   $$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$$
3. $\int \tan^2(x)\sec(x)\ dx = \int (\sec^2(x) - 1)\sec(x)\ dx$
   $$= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$$
4. $\int \sec^2(x)\tan(x)\ dx$
   $$= \frac{1}{2}\tan^2(x) + C.$$
5. $\int \cot^3(x)\ dx$
   $$= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$

   $= \tan(x) - x + C.$

2. $\int \tan^3(x)\ dx = \int (\sec^2(x) - 1)\tan(x)\ dx$

   $= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$

3. $\int \tan^2(x)\sec(x)\ dx = \int (\sec^2(x) - 1)\sec(x)\ dx$

   $= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$

4. $\int \sec^2(x)\tan(x)\ dx$ Let $u = \tan(x)$, so $du = \sec^2(x)$:

   $= \frac{1}{2}\tan^2(x) + C.$

5. $\int \cot^3(x)\ dx$

   $= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$

# You try:

Compute the following antiderivatives:

1. $\int \tan^2(x)\ dx = \int \sec^2(x) - 1\ dx$

$$= \tan(x) - x + C.$$

2. $\int \tan^3(x)\ dx = \int (\sec^2(x) - 1) \tan(x)\ dx$

$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.$$

3. $\int \tan^2(x) \sec(x)\ dx = \int (\sec^2(x) - 1) \sec(x)\ dx$

$$= \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)| + C.$$

4. $\int \sec^2(x) \tan(x)\ dx$ Let $u = \tan(x)$, so $du = \sec^2(x)$:

$$= \frac{1}{2}\tan^2(x) + C.$$

5. $\int \cot^3(x)\ dx$ Using $1 + \cot^2(x) = \csc^2(x)$,

$$\int \cot^3(x)\ dx = \int (\csc^2(x) - 1) \cot(x)\ dx$$

$$= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + C.$$