Today: 6.1 Integration by parts

Warm up: Differentiate the following functions

(1) $x \ln(x)$, (2) $\cosh(x)\sinh(x)$, (3) $\cos(x)\sin(x)$,

(4) $\sqrt{x}\sinh^{-1}(x)$, (5) $3^x x^3$, (6) $(1+x^2)\tan^{-1}(x)$,

Integration by parts: undoing product rule

Recall, product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides, we get

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$

How this helps: Calculate

$$\int \underbrace{x}_{f} \underbrace{\cos(x)}_{g'} \, dx.$$

Wouldn't it be great if we could get rid of x?

$$x\sin(x) = \int 1\cdot\sin(x)\,dx + \int x\cos(x)\,dx = -\cos(x) + \int x\cos(x)\,dx.$$

So $\int x\cos(x)\,dx = x\sin(x) + \cos(x)$.

Integration by parts: undoing product rule

Integrating both sides of $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$ gives

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

So

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Revisiting the previous example: $\int x\cos(x)\,dx$.
Start by picking

$f(x)$ so that (usually) $f'(x)$ is simpler than $f(x)$,

and $g'(x)$ so that we can compute $\int g'(x)\,dx$.

$f(x)$: x	$g(x)$: $\sin(x)$
$f'(x)$: 1	$g'(x)$: $\cos(x)$

Then compute $f'(x)$ and $g(x)$.
Finally, plug in and hope we're left with something we can compute:

$$\int x\cos(x)\,dx = x\cdot\sin(x) - \int 1\cdot\cos(x)\,dx \checkmark$$

You try

$$\boxed{\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx.}$$

Compute the integral

$$\int xe^x\ dx$$

by letting

$$f(x) = x \quad \text{and} \quad g'(x) = e^x.$$

(Compute $f'(x)$ and $g(x) = \int g'(x)\ dx$, and then plug into the formula)

You try: Do it twice

$$\boxed{\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx.}$$

Compute $\displaystyle\int x^2 \cos(x)\ dx.$

Start by letting

$$f(x) = x^2 \quad \text{and} \quad g'(x) = \cos(x).$$

When you get to computing $\int f'(x)g(x)\ dx$, you'll need to use integration by parts again.
Answer: $(x^2 - 2)\sin(x) + 2x\cos(x) + C$

A note on notation: In the book, they use u and dv in place of $f(x)$ and $g'(x)$. You can use either.

You try: Something out of nothing

$$\boxed{\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx.}$$

Compute

$$\int \ln(x)\ dx$$

using integration by parts by letting

$$f(x) = \ln(x) \quad \text{and} \quad g'(x) = 1.$$

Answer: $x(\ln(x) - 1) + C$

Note: This is a common technique for computing the integral of inverse functions, like $\sin^{-1}(x)$, $\tan^{-1}(x)$, $\sinh^{-1}(x)$, etc..

Something else out of nothing, u-sub edition

$$\text{Compute} \quad \int e^{\sqrt{x}}\, dx.$$

We could try letting $f(x) = e^{\sqrt{x}}$ and $g'(x) = 1$, like before.
So $f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$ and $g(x) = x$. Thus

$$\int e^{\sqrt{x}}\, dx = x e^{\sqrt{x}} - \tfrac{1}{2} \underbrace{\int \sqrt{x} e^{\sqrt{x}}\, dx}_{\text{not a lot better...}}.$$

Instead, we can start with a u-substitution:
Let $u = \sqrt{x}$. So $du = \frac{1}{2\sqrt{x}} dx = \frac{1}{2u} dx$.
So $2u\, du = dx$, giving

$$\int e^{\sqrt{x}}\, dx = \int 2u e^u du.$$

Finish using integration by parts like before!

A cyclic example: Compute $\int e^x \cos(x)\, dx$.

Let $f(x) = e^x$ and $g'(x) = \cos(x)$. So $f'(x) = e^x$ and $g(x) = \sin(x)$.
Thus

$$\int e^x \cos(x)\, dx = e^x \sin(x) - \int e^x \sin(x)\, dx.$$

Now, to compute $\int e^x \sin(x)\, dx$:
Let $f(x) = e^x$ and $g'(x) = \sin(x)$. So $f'(x) = e^x$ and $g(x) = -\cos(x)$.
Thus

$$\int e^x \sin(x)\, dx = -e^x \cos(x) - \int e^x(-\cos(x))\, dx$$

Putting these together, we have

$$\int e^x \cos(x)\, dx = e^x \sin(x) - \left(-e^x \cos(x) + \int e^x \cos(x)\right).$$

$$\text{So} \qquad 2\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) + C.$$

And thus

$$\int e^x \cos(x)\, dx = \tfrac{1}{2} e^x(\sin(x) + \cos(x)) + D.$$

You try:

Compute the following antiderivatives using integration by parts.

1. $\int x \sec^2(x)\ dx$ Answer: $x\tan(x) + \ln(\cos(x)) + C$
2. $\int \sin^{-1}(x)\ dx$ Answer: $x\sin^{-1}(x) + \sqrt{1-x^2} + C$
3. $\int x\ln(x)\ dx$ Answer: $x^2(2\ln(x) - 1)/4 + C$

   (Try first by letting $f(x) = x$, and try again letting $f(x) = \ln(x)$)
4. $\int \cos^2(x)\ dx$ Answer: $\frac{1}{2}(x + \sin(x)\cos(x)) + C$

   (Try letting $f(x) = \cos(x)$ and $g'(x) = \cos(x)$, and then using $\sin^2(x) = 1 - \cos^2(x)$. Then look back at the $e^x\cos(x)$ example.)