

Today: 5.8 L'Hospital's rule continued.

Recall, L'Hospital's rule says that if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ or } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \pm\infty,$$

then

$$\lim_{x \rightarrow a} f(x)/g(x) = \lim_{x \rightarrow a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \rightarrow \pm\infty$.

Warm up: Use whatever methods you have at your disposal to calculate the following limits.

$$(1) \lim_{x \rightarrow 0^+} \frac{\sin(x)}{1 - \cos(x)}, \quad (2) \lim_{x \rightarrow 0} \frac{3^x - e^x}{x}, \quad (3) \lim_{x \rightarrow 0} \frac{2x^2 + x}{3x^2 + 1},$$

$$(4) \lim_{x \rightarrow \infty} x^{-\ln(x)}, \quad (5) \lim_{x \rightarrow \infty} \tan^{-1}(e^x/x), \quad (6) \lim_{x \rightarrow -\infty} \tan^{-1}(e^x/x),$$

$$(7) \lim_{x \rightarrow \infty} e^{-x} \tan^{-1}(x), \quad (8) \lim_{x \rightarrow 0^-} \frac{x}{|x|}, \quad (9) \lim_{x \rightarrow \infty} \frac{\ln(\sqrt{x})}{x^2}.$$

Answers: ∞ , $\ln(3) - 1$, 0 , 0 , $\pi/2$, 0 , 0 , -1 , 0 .

Answers

1. $\lim_{x \rightarrow 0^+} \frac{\sin(x)}{1 - \cos(x)} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x \rightarrow 0} \frac{3^x - e^x}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x \rightarrow 0} \frac{2x^2 + x}{3x^2 + 1} = 0/1 = 0$
4. $\lim_{x \rightarrow \infty} x^{-\ln(x)} = \lim_{x \rightarrow \infty} 1/x^{\ln(x)} = 0$
5. $\lim_{x \rightarrow \infty} \tan^{-1}(e^x/x)$: Let $u = e^x/x$. Then as $x \rightarrow \infty$,

$$\lim_{x \rightarrow \infty} u = \lim_{x \rightarrow \infty} e^x/x \stackrel{\text{L'H}}{=} e^x/1 = \infty.$$

Thus

$$\lim_{x \rightarrow \infty} \tan^{-1}(e^x/x) = \lim_{u \rightarrow \infty} \tan^{-1}(u) = \pi/2.$$

Answers

6. $\lim_{x \rightarrow -\infty} \tan^{-1}(e^x/x)$: As $x \rightarrow -\infty$, $e^x/x \rightarrow 0$, so

$$\lim_{x \rightarrow -\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

7. $\lim_{x \rightarrow \infty} e^{-x} \tan^{-1}(x) = 0$, since $\tan^{-1}(x)$ stays between $\pm\pi/2$ and $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$.
8. $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x}{-x} = -1$.
9. $\lim_{x \rightarrow \infty} \frac{\ln(\sqrt{x})}{x^2} = \lim_{x \rightarrow \infty} \frac{\ln(x)}{2x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{1/x}{4x} = 0$.

exponentials \gg powers \gg logarithms

Question: How does e^x grow versus x^a ?

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^x}{x} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty \\ \lim_{x \rightarrow \infty} \frac{e^x}{x^2} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty \\ \lim_{x \rightarrow \infty} \frac{e^x}{x^3} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty \\ \lim_{x \rightarrow \infty} \frac{e^x}{x^{3/2}} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{e^x}{\frac{3}{2}x^{1/2}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{2}{3} \frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x \rightarrow \infty} \frac{4}{3} e^x x^{1/2} = \infty\end{aligned}$$

For any a , there is some n for which $\frac{d^n}{dx^n} x^a$ is some constant times x^{a-n} such that $a - n \leq 0$. So

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^a} = \infty \quad \text{for all } a!$$

You try: For what a does $x^a / \ln(x)$ approach ∞ as $x \rightarrow \infty$?

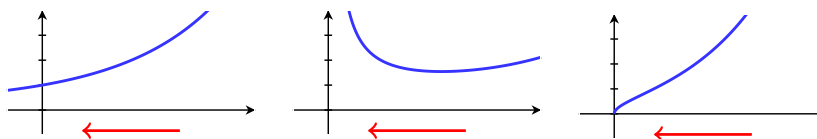
Other indeterminate forms

Our first two indeterminate forms were

(1) f/g if $f, g \rightarrow \pm\infty$ and (2) f/g if $f, g \rightarrow 0$

(called **type ∞/∞** and **type $0/0$**). They're **indeterminate** since any number of things can happen. For example, as $x \rightarrow 0^+$,

$$\frac{e^x - 1}{x} \rightarrow 1 \quad \frac{e^x - 1}{x^2} \rightarrow \infty \quad \frac{e^x - 1}{\sqrt{x}} \rightarrow 0$$



To this list, we add

(3) fg if $f \rightarrow 0$ and $g \rightarrow \pm\infty$

Notice, if $g(x) \rightarrow 0^\pm$, then $1/g(x) \rightarrow \pm\infty$.

Example: Compute $\lim_{x \rightarrow 0^+} x \ln(x)$. We rewrite this as

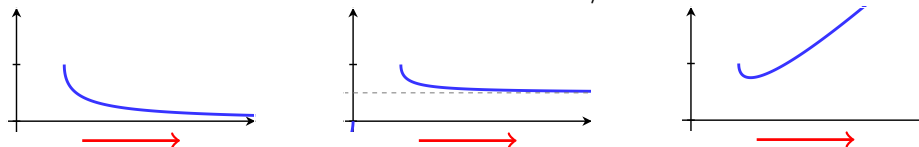
$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-1}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-x^{-2}} = - \lim_{x \rightarrow 0^+} x = 0.$$

Other indeterminate forms

(4) $f - g$ if $f, g \rightarrow \infty$ (called **type $\infty - \infty$**)

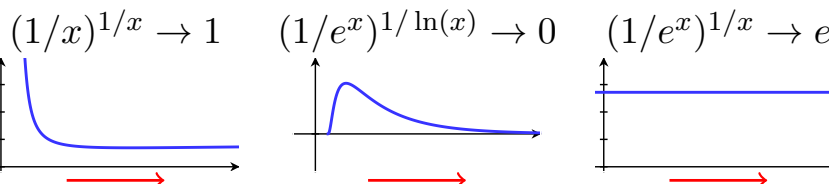
For example, as $x \rightarrow \infty$,

$$x - \sqrt{x^2 - 1} \rightarrow 0 \quad x - \sqrt{x^2 - x} \rightarrow 1/2 \quad x - \sqrt{x - 1} \rightarrow \infty$$



(5) f^g if $f, g \rightarrow 0$ (called **type 0^0**)

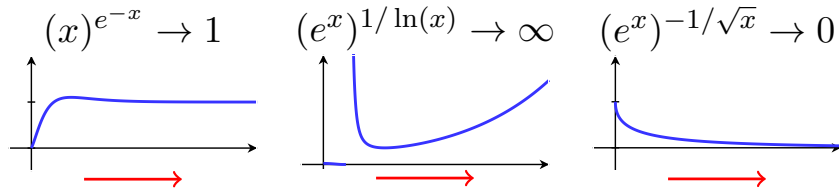
For example, as $x \rightarrow \infty$,



Other indeterminate forms

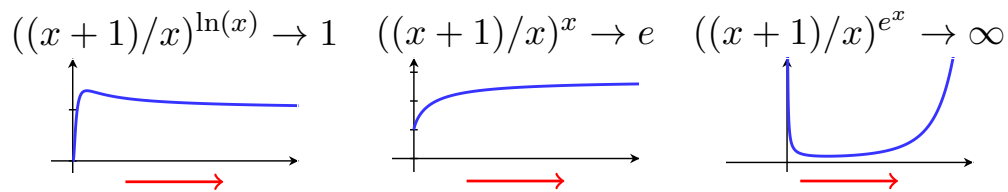
(6) f^g if $f \rightarrow \infty$ and $g \rightarrow 0$ (called type ∞^0)

For example, as $x \rightarrow \infty$,



(7) f^g if $f \rightarrow 1$ and $g \rightarrow \infty$ (called type 1^∞)

For example, as $x \rightarrow \infty$,



You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is. |

1. $\lim_{x \rightarrow \infty} x - \ln(x)$
2. $\lim_{x \rightarrow 0^+} x - \ln(x)$
3. $\lim_{x \rightarrow \infty} x^x$
4. $\lim_{x \rightarrow 0^+} x^x$
5. $\lim_{x \rightarrow \infty} (1/x)^x$
6. $\lim_{x \rightarrow 0^+} (1 + \sin(x))^{\cot(x)}$
7. $\lim_{x \rightarrow \pi/2^+} \sec(x) - \tan(x)$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x \rightarrow a} G(x) = L$, then

$$\lim_{x \rightarrow a} F(G(x)) = f\left(\lim_{x \rightarrow a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x \rightarrow a} G(x)\right) = \lim_{x \rightarrow a} \ln(G(x)).$$

Since $\ln(x)$ is invertible over the positive real line, if I can compute the limit of $\ln(G(x))$, then I can solve for the limit of $G(x)$.

Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x) \ln(f(x))$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x \rightarrow 0^+} x^x$.

This has indeterminate form 0^0 .

Let $L = \lim_{x \rightarrow 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x \rightarrow 0^+} x^x\right) = \lim_{x \rightarrow 0^+} \ln(x^x) = \lim_{x \rightarrow 0^+} x \ln(x).$$

Now we've changed this into the indeterminate form $0 \cdot \infty$, which we know how to solve! We saw before that

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-1}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \rightarrow 0^+} -x = 0.$$

So

$$\ln(L) = 0, \quad \text{implying } L = e^0 = 1.$$

So $\boxed{\lim_{x \rightarrow 0^+} x^x = 1}$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x \rightarrow a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0 , ∞^0 , or 1^∞).

Step 1: Let $L = \lim_{x \rightarrow a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x \rightarrow a} \ln(f(x)^{g(x)}) = \lim_{x \rightarrow a} g(x) \ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x \rightarrow a} g(x) \ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L .

You try: Calculate the following limits.

$$(1) \lim_{x \rightarrow \infty} x^{e^{-x}}, \quad (2) \lim_{x \rightarrow \infty} (e^x)^{1/\ln(x)}$$

$$(3) \lim_{x \rightarrow 0^+} (1 + \sin(x))^{\cot(x)} \quad (4) \lim_{x \rightarrow 0^+} (1 + \sin(3x))^{\cot(x)}$$

(recall, $\cos(0) = 1$, $\sin(0) = 0$)

Answers: 1, ∞ , e , e^3 .

Alternate solution for $\lim_{x \rightarrow \infty} (e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x \rightarrow \infty} (e^x)^{1/\ln(x)} = \lim_{x \rightarrow \infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x \rightarrow \infty} e^{x/\ln(x)} = \exp\left(\lim_{x \rightarrow \infty} x/\ln(x)\right)$$

$$\stackrel{\text{L'H}}{=} \exp\left(\lim_{x \rightarrow \infty} 1/(1/x)\right) = \exp\left(\lim_{x \rightarrow \infty} x\right) = \infty.$$

Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.

Example: $L = \lim_{x \rightarrow 0^+} \csc(x) - \cot(x)$. Note

$$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{1 - \cos(x)}{\sin(x)}.$$

So by L'Hospital,

$$L = \lim_{x \rightarrow 0^+} \frac{1 - \cos(x)}{\sin(x)} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\sin(x)}{\cos(x)} = 0.$$

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2. Use identities like $(a - b)(a + b) = a^2 - b^2$ to get rid of square roots.

Example: $L = \lim_{x \rightarrow \infty} x - \sqrt{x^2 - x}.$ Note

$$\begin{aligned} x - \sqrt{x^2 - x} &= (x - \sqrt{x^2 - x}) \left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} \right) \\ &= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}. \end{aligned}$$

So

$$L = \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}} \left(\frac{1/x}{1/x} \right) = \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 - 1/x}} = 1/2$$

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Example: $\lim_{x \rightarrow \infty} x - \sqrt{x^2 - x} = \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}} = 1/2$

3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.

Example: $L = \lim_{x \rightarrow \infty} \ln(x) - x^2.$ Start with

$$e^L = \exp \left(\lim_{x \rightarrow \infty} \ln(x) - x^2 \right) = \lim_{x \rightarrow \infty} e^{\ln(x) - x^2}.$$

Then since $e^{\ln(x) - x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$, we have

$$e^L = \lim_{x \rightarrow \infty} x/e^{x^2} = 0, \quad \text{so } L = -\infty.$$

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \rightarrow 0^+} \sin^{-1}(x)/x$ Ans: 1

2. $\lim_{x \rightarrow 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$ Ans: 1/2

3. $\lim_{x \rightarrow \infty} x \sin(\pi/x)$ Ans: π

4. $\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$ Ans: 3

5. $\lim_{x \rightarrow \infty} x^{\ln(2)/(1+\ln(x))}$ Ans: 2

6. $\lim_{x \rightarrow 0} \frac{\tan(x)}{\tanh(x)}$ Ans: 1

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.