Today: 5.8 L'Hospital's rule continued.

Recall, L'Hospital's rule says that if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \to \pm\infty$.

Warm up: Use whatever methods you have at your disposal to calculate the following limits.

(1) $\displaystyle\lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)}$, (2) $\displaystyle\lim_{x\to 0} \frac{3^x - e^x}{x}$, (3) $\displaystyle\lim_{x\to 0} \frac{2x^2+x}{3x^2+1}$,

(4) $\displaystyle\lim_{x\to\infty} x^{-\ln(x)}$, (5) $\displaystyle\lim_{x\to\infty} \tan^{-1}(e^x/x)$, (6) $\displaystyle\lim_{x\to-\infty} \tan^{-1}(e^x/x)$,

(7) $\displaystyle\lim_{x\to\infty} e^{-x}\tan^{-1}(x)$, (8) $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|}$, (9) $\displaystyle\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}$.

Today: 5.8 L'Hospital's rule continued.

Recall, L'Hospital's rule says that if f and g are differentiable, $g'(x) \neq 0$ near a (but $g'(a) = 0$ is ok), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

Same goes for one-sided limits and $x \to \pm\infty$.

Warm up: Use whatever methods you have at your disposal to calculate the following limits.

(1) $\displaystyle\lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)}$, (2) $\displaystyle\lim_{x\to 0} \frac{3^x - e^x}{x}$, (3) $\displaystyle\lim_{x\to 0} \frac{2x^2+x}{3x^2+1}$,

(4) $\displaystyle\lim_{x\to\infty} x^{-\ln(x)}$, (5) $\displaystyle\lim_{x\to\infty} \tan^{-1}(e^x/x)$, (6) $\displaystyle\lim_{x\to-\infty} \tan^{-1}(e^x/x)$,

(7) $\displaystyle\lim_{x\to\infty} e^{-x}\tan^{-1}(x)$, (8) $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|}$, (9) $\displaystyle\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}$.

Answers: ∞, $\ln(3)-1$, 0, 0, $\pi/2$, 0, 0, -1, 0.

Answers

1. $\lim_{x \to 0^+} \frac{\sin(x)}{1 - \cos(x)}$
2. $\lim_{x \to 0} \frac{3^x - e^x}{x}$
3. $\lim_{x \to 0} \frac{2x^2 + x}{3x^2 + 1}$
4. $\lim_{x \to \infty} x^{-\ln(x)}$
5. $\lim_{x \to \infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x\to 0^+} \frac{\sin(x)}{1-\cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x\to 0} \frac{3^x - e^x}{x}$
3. $\lim_{x\to 0} \frac{2x^2 + x}{3x^2 + 1}$
4. $\lim_{x\to \infty} x^{-\ln(x)}$
5. $\lim_{x\to \infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x \to 0^+} \frac{\sin(x)}{1 - \cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x \to 0} \frac{3^x - e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x \to 0} \frac{2x^2 + x}{3x^2 + 1}$
4. $\lim_{x \to \infty} x^{-\ln(x)}$
5. $\lim_{x \to \infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x \to 0^+} \frac{\sin(x)}{1 - \cos(x)} \stackrel{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x \to 0} \frac{3^x - e^x}{x} \stackrel{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x \to 0} \frac{2x^2 + x}{3x^2 + 1} = 0/1 = 0$
4. $\lim_{x \to \infty} x^{-\ln(x)}$
5. $\lim_{x \to \infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x\to 0^+} \dfrac{\sin(x)}{1-\cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \dfrac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x\to 0} \dfrac{3^x - e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \dfrac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x\to 0} \dfrac{2x^2 + x}{3x^2 + 1} = 0/1 = 0$
4. $\lim_{x\to\infty} x^{-\ln(x)} = \lim_{x\to\infty} 1/x^{\ln(x)} = 0$
5. $\lim_{x\to\infty} \tan^{-1}(e^x/x)$

Answers

1. $\lim_{x \to 0^+} \frac{\sin(x)}{1-\cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x \to 0} \frac{3^x - e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x \to 0} \frac{2x^2 + x}{3x^2 + 1} = 0/1 = 0$
4. $\lim_{x \to \infty} x^{-\ln(x)} = \lim_{x \to \infty} 1/x^{\ln(x)} = 0$
5. $\lim_{x \to \infty} \tan^{-1}(e^x/x)$: Let $u = e^x/x$. Then as $x \to \infty$,

$$\lim_{x \to \infty} u = \lim_{x \to \infty} e^x/x \overset{\text{L'H}}{=\!=} e^x/1 = \infty.$$

Answers

1. $\lim_{x\to 0^+} \dfrac{\sin(x)}{1-\cos(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \dfrac{\cos(x)}{\sin(x)} = \infty$
2. $\lim_{x\to 0} \dfrac{3^x - e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to 0} \dfrac{\ln(3)3^x - e^x}{1} = \ln(3) - 1$
3. $\lim_{x\to 0} \dfrac{2x^2 + x}{3x^2 + 1} = 0/1 = 0$
4. $\lim_{x\to\infty} x^{-\ln(x)} = \lim_{x\to\infty} 1/x^{\ln(x)} = 0$
5. $\lim_{x\to\infty} \tan^{-1}(e^x/x)$: Let $u = e^x/x$. Then as $x \to \infty$,

$$\lim_{x\to\infty} u = \lim_{x\to\infty} e^x/x \overset{\text{L'H}}{=\!=} e^x/1 = \infty.$$

Thus

$$\lim_{x\to\infty} \tan^{-1}(e^x/x) = \lim_{u\to\infty} \tan^{-1}(u) = \pi/2.$$

Answers

6. $\lim_{x \to -\infty} \tan^{-1}(e^x/x)$

7. $\lim_{x \to \infty} e^{-x} \tan^{-1}(x)$

8. $\lim_{x \to 0^-} \frac{x}{|x|}$.

9. $\lim_{x \to \infty} \frac{\ln(\sqrt{x})}{x^2}$

Answers

6. $\lim_{x\to-\infty} \tan^{-1}(e^x/x)$: As $x \to -\infty$, $e^x/x \to 0$, so

$$\lim_{x\to-\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

7. $\lim_{x\to\infty} e^{-x} \tan^{-1}(x)$

8. $\lim_{x\to 0^-} \frac{x}{|x|}$.

9. $\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}$

Answers

6. $\lim_{x \to -\infty} \tan^{-1}(e^x/x)$: As $x \to -\infty$, $e^x/x \to 0$, so

$$\lim_{x \to -\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

7. $\lim_{x \to \infty} e^{-x} \tan^{-1}(x) = 0$, since $\tan^{-1}(x)$ stays between $\pm\pi/2$ and $e^{-x} \to 0$ as $x \to \infty$.
8. $\lim_{x \to 0^-} \frac{x}{|x|}$.
9. $\lim_{x \to \infty} \frac{\ln(\sqrt{x})}{x^2}$

Answers

6. $\lim_{x \to -\infty} \tan^{-1}(e^x/x)$: As $x \to -\infty$, $e^x/x \to 0$, so

$$\lim_{x \to -\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

7. $\lim_{x \to \infty} e^{-x} \tan^{-1}(x) = 0$, since $\tan^{-1}(x)$ stays between $\pm\pi/2$ and $e^{-x} \to 0$ as $x \to \infty$.
8. $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1.$
9. $\lim_{x \to \infty} \frac{\ln(\sqrt{x})}{x^2}$

Answers

6. $\lim_{x\to-\infty} \tan^{-1}(e^x/x)$: As $x \to -\infty$, $e^x/x \to 0$, so

$$\lim_{x\to-\infty} \tan^{-1}(e^x/x) = \tan^{-1}(0) = 0.$$

7. $\lim_{x\to\infty} e^{-x} \tan^{-1}(x) = 0$, since $\tan^{-1}(x)$ stays between $\pm\pi/2$ and $e^{-x} \to 0$ as $x \to \infty$.
8. $\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = -1.$
9. $\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2} = \lim_{x\to\infty} \frac{\ln(x)}{2x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{1/x}{4x} = 0.$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x}$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}x^{1/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty}\frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{2}{3}\frac{e^x}{\frac{1}{2}x^{-1/2}}$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty}\frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty}\frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty}\frac{2}{3}\frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{4}{3}e^x x^{1/2} = \infty$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{2}{3}\frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{4}{3}e^x x^{1/2} = \infty$$

For any a, there is some n for which $\frac{d^n}{dx^n}x^a$ is some constant times x^{a-n} such that $a - n \leq 0$. So

$$\lim_{x\to\infty} \frac{e^x}{x^a} = \infty \quad \text{for all } a!$$

exponentials $\gg$ powers $\gg$ logarithms

Question: How does e^x grow versus x^a?

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x^{3/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}x^{1/2}} \overset{\text{L'H}}{=\!=} \lim_{x\to\infty} \frac{2}{3}\frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{4}{3}e^x x^{1/2} = \infty$$

For any a, there is some n for which $\frac{d^n}{dx^n}x^a$ is some constant times x^{a-n} such that $a - n \leq 0$. So

$$\lim_{x\to\infty} \frac{e^x}{x^a} = \infty \quad \text{for all } a!$$

You try: For what a does $x^a/\ln(x)$ approach ∞ as $x \to \infty$?

Other indeterminate forms

Our first two indeterminate forms were

$$(1)\ f/g \quad \text{if} \quad f, g \to \pm\infty \quad \text{and} \quad (2)\ f/g \quad \text{if} \quad f, g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen.

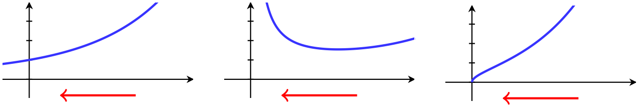
Other indeterminate forms

Our first two indeterminate forms were

$$(1)\ f/g \quad \text{if} \quad f, g \to \pm\infty \quad \text{and} \quad (2)\ f/g \quad \text{if} \quad f, g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x - 1}{x} \to 1 \qquad \frac{e^x - 1}{x^2} \to \infty \qquad \frac{e^x - 1}{\sqrt{x}} \to 0$$



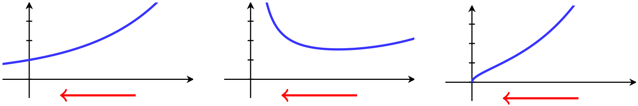
Other indeterminate forms

Our first two indeterminate forms were

$$(1)\ f/g \quad \text{if} \quad f, g \to \pm\infty \quad \text{and} \quad (2)\ f/g \quad \text{if} \quad f, g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x - 1}{x} \to 1 \qquad \frac{e^x - 1}{x^2} \to \infty \qquad \frac{e^x - 1}{\sqrt{x}} \to 0$$



To this list, we add

$$(3)\ fg \text{ if } \quad f \to 0 \text{ and } g \to \pm\infty$$

Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.

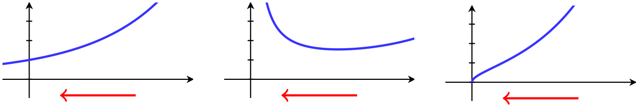
Other indeterminate forms

Our first two indeterminate forms were

$$(1)\ f/g \quad \text{if} \quad f, g \to \pm\infty \quad \text{and} \quad (2)\ f/g \quad \text{if} \quad f, g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x - 1}{x} \to 1 \qquad \frac{e^x - 1}{x^2} \to \infty \qquad \frac{e^x - 1}{\sqrt{x}} \to 0$$



To this list, we add

$$(3)\ fg \text{ if} \quad f \to 0 \text{ and } g \to \pm\infty$$

Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.
Example: Compute $\lim_{x \to 0^+} x \ln(x)$.

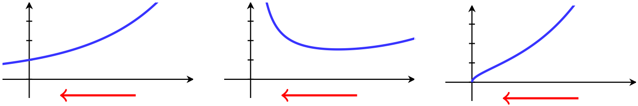
Other indeterminate forms

Our first two indeterminate forms were

(1) f/g if $f, g \to \pm\infty$ and (2) f/g if $f, g \to 0$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x - 1}{x} \to 1 \qquad \frac{e^x - 1}{x^2} \to \infty \qquad \frac{e^x - 1}{\sqrt{x}} \to 0$$



To this list, we add

(3) fg if $f \to 0$ and $g \to \pm\infty$

Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.
Example: Compute $\lim_{x\to 0^+} x\ln(x)$. We rewrite this as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}}$$

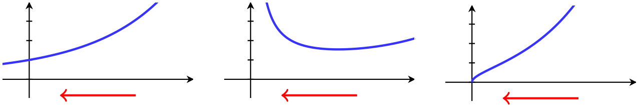
Other indeterminate forms

Our first two indeterminate forms were

$$\textcolor{blue}{(1)}\ f/g \quad \text{if} \quad f,g \to \pm\infty \quad \text{and} \quad \textcolor{blue}{(2)}\ f/g \quad \text{if} \quad f,g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x-1}{x} \to 1 \qquad \frac{e^x-1}{x^2} \to \infty \qquad \frac{e^x-1}{\sqrt{x}} \to 0$$



To this list, we add

$$\textcolor{blue}{(3)}\ fg \text{ if } \quad f \to 0 \text{ and } g \to \pm\infty$$

Notice, if $g(x) \to 0^{\pm}$, then $1/g(x) \to \pm\infty$.

Example: Compute $\lim_{x\to 0^+} x\ln(x)$. We rewrite this as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}$$

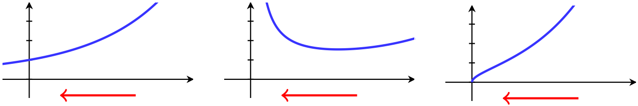
Other indeterminate forms

Our first two indeterminate forms were

$$(1)\ f/g \quad \text{if} \quad f,g \to \pm\infty \quad \text{and} \quad (2)\ f/g \quad \text{if} \quad f,g \to 0$$

(called type ∞/∞ and type $0/0$). They're indeterminate since any number of things can happen. For example, as $x \to 0^+$,

$$\frac{e^x-1}{x} \to 1 \qquad \frac{e^x-1}{x^2} \to \infty \qquad \frac{e^x-1}{\sqrt{x}} \to 0$$



To this list, we add

$$(3)\ fg \text{ if } \quad f \to 0 \text{ and } g \to \pm\infty$$

Notice, if $g(x) \to 0^\pm$, then $1/g(x) \to \pm\infty$.

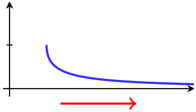
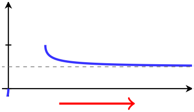
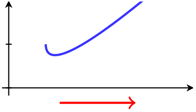
Example: Compute $\lim_{x\to 0^+} x\ln(x)$. We rewrite this as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = -\lim_{x\to 0^+} x = 0.$$

Other indeterminate forms

(4) $f - g$ if $f, g \to \infty$ (called type $\infty - \infty$)
For example, as $x \to \infty$,

$$x - \sqrt{x^2 - 1} \to 0 \qquad x - \sqrt{x^2 - x} \to 1/2 \qquad x - \sqrt{x - 1} \to \infty$$

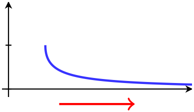




Other indeterminate forms

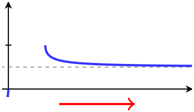
(4) $f - g$ if $f, g \to \infty$ (called type $\infty - \infty$)

For example, as $x \to \infty$,

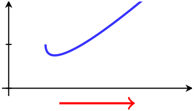
$x - \sqrt{x^2 - 1} \to 0$



$x - \sqrt{x^2 - x} \to 1/2$



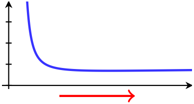
$x - \sqrt{x - 1} \to \infty$



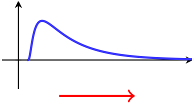
(5) f^g if $f, g \to 0$ (called type 0^0)

For example, as $x \to \infty$,

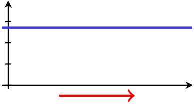
$(1/x)^{1/x} \to 1$



$(1/e^x)^{1/\ln(x)} \to 0$



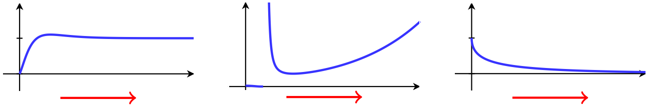
$(1/e^x)^{1/x} \to e$



Other indeterminate forms

(6) f^g if $f \to \infty$ and $g \to 0$ (called type ∞^0)
For example, as $x \to \infty$,

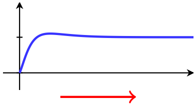
$(x)^{e^{-x}} \to 1$ $\quad$ $(e^x)^{1/\ln(x)} \to \infty$ $\quad$ $(e^x)^{-1/\sqrt{x}} \to 0$



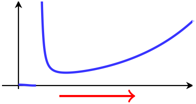
Other indeterminate forms

(6) f^g if $f \to \infty$ and $g \to 0$ (called type ∞^0)
For example, as $x \to \infty$,

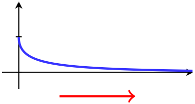
$(x)^{e^{-x}} \to 1$



$(e^x)^{1/\ln(x)} \to \infty$

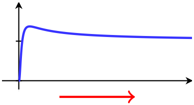


$(e^x)^{-1/\sqrt{x}} \to 0$

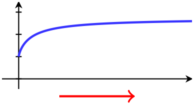


(7) f^g if $f \to 1$ and $g \to \infty$ (called type 1^∞)
For example, as $x \to \infty$,

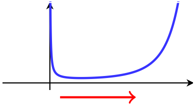
$((x+1)/x)^{\ln(x)} \to 1$



$((x+1)/x)^{x} \to e$



$((x+1)/x)^{e^x} \to \infty$



You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is. I

1. $\lim_{x \to \infty} x - \ln(x)$
2. $\lim_{x \to 0^+} x - \ln(x)$
3. $\lim_{x \to \infty} x^x$
4. $\lim_{x \to 0^+} x^x$
5. $\lim_{x \to \infty} (1/x)^x$
6. $\lim_{x \to 0^+} (1 + \sin(x))^{\cot(x)}$
7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$

You try:

To summarize, we have 7 indeterminate form types:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad \text{and} \quad 1^\infty.$$

For each of the following limits, decide if the limit is an indeterminate form. If so, identify which indeterminate form it is. I

1. $\lim_{x \to \infty} x - \ln(x)$ Ans: type $\infty - \infty$
2. $\lim_{x \to 0^+} x - \ln(x) = 0 - (-\infty) = \infty$ Ans: not indet
3. $\lim_{x \to \infty} x^x = \infty$ Ans: not indet
4. $\lim_{x \to 0^+} x^x$ Ans: type 0^0
5. $\lim_{x \to \infty} (1/x)^x = 0$ Ans: not indet! (see 5.8#52)
6. $\lim_{x \to 0^+} (1 + \sin(x))^{\cot(x)}$ Ans: type 1^∞
7. $\lim_{x \to \pi/2^+} \sec(x) - \tan(x)$ Ans: type $\infty - \infty$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x\to a} G(x)\right) = \lim_{x\to a} \ln(G(x)).$$

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x\to a} G(x)\right) = \lim_{x\to a} \ln(G(x)).$$

Since $\ln(x)$ is invertible over the positive real line, if I can compute the limit of $\ln(G(x))$, then I can solve for the limit of $G(x)$.

Solving exponential indeterminate forms

Recall the property of limits, that if $F(x)$ is continuous at L and $\lim_{x\to a} G(x) = L$, then

$$\lim_{x\to a} F(G(x)) = f\left(\lim_{x\to a} G(x)\right) = F(L).$$

In particular, since $F(x) = \ln(x)$ is continuous,

$$\ln\left(\lim_{x\to a} G(x)\right) = \lim_{x\to a} \ln(G(x)).$$

Since $\ln(x)$ is invertible over the positive real line, if I can compute the limit of $\ln(G(x))$, then I can solve for the limit of $G(x)$.

Why do I like this? Logarithms turn exponentials into products!

$$\ln(f(x)^{g(x)}) = g(x)\ln(f(x))$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x \to 0^+} x^x$.

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.

Solving exponential indeterminate forms

Example: Compute $\lim_{x \to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x \to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x \to 0^+} x^x\right) = \lim_{x \to 0^+} \ln(x^x)$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x \to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x \to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x \to 0^+} x^x\right) = \lim_{x \to 0^+} \ln(x^x) = \lim_{x \to 0^+} x \ln(x).$$

Now we've changed this into the indeterminate form $0 \cdot \infty$, which we know how to solve!

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0 \cdot \infty$, which we know how to solve! We saw before that

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0 \cdot \infty$, which we know how to solve! We saw before that

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

So

$$\ln(L) = 0$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0\cdot\infty$, which we know how to solve! We saw before that

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

So

$$\ln(L) = 0, \quad \text{implying } L = e^0 = 1.$$

Solving exponential indeterminate forms

Example: Compute $\lim_{x\to 0^+} x^x$.
This has indeterminate form 0^0.
Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln(L) = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x).$$

Now we've changed this into the indeterminate form $0\cdot\infty$, which we know how to solve! We saw before that

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

So

$$\ln(L) = 0, \quad \text{implying } L = e^0 = 1.$$

So $\boxed{\lim_{x\to 0^+} x^x = 1}$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L.

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L.

You try: Calculate the following limits.

$$(1)\ \lim_{x\to\infty} x^{e^{-x}}, \quad (2)\ \lim_{x\to\infty} (e^x)^{1/\ln(x)}$$

$$(3)\ \lim_{x\to 0^+} (1+\sin(x))^{\cot(x)} \quad (4)\ \lim_{x\to 0^+} (1+\sin(3x))^{\cot(x)}$$

(recall, $\cos(0) = 1$, $\sin(0) = 0$)

Solving exponential indeterminate forms

Say you want to compute $\lim_{x\to a} f(x)^{g(x)}$, where $f(x)^{g(x)}$ approaches one of the three indeterminate forms (0^0, ∞^0, or 1^∞).

Step 1: Let $L = \lim_{x\to a} f(x)^{g(x)}$. Then

$$\ln(L) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a} g(x)\ln(f(x)).$$

Step 2: Simplify, and if necessary, use L'Hospital's rule to calculate $\lim_{x\to a} g(x)\ln(f(x)) = M$.

Step 3: Finally, $\ln(L) = M$ implies $L = e^M$ solves for L.

You try: Calculate the following limits.

$$(1)\ \lim_{x\to\infty} x^{e^{-x}}, \quad (2)\ \lim_{x\to\infty} (e^x)^{1/\ln(x)}$$

$$(3)\ \lim_{x\to 0^+} (1+\sin(x))^{\cot(x)} \quad (4)\ \lim_{x\to 0^+} (1+\sin(3x))^{\cot(x)}$$

(recall, $\cos(0) = 1$, $\sin(0) = 0$)

Answers: 1, ∞, e, e^3.

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\overset{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right)$$

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\stackrel{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right) = \exp\left(\lim_{x\to\infty} x\right)$$

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\overset{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right) = \exp\left(\lim_{x\to\infty} x\right) = \infty.$$

Alternate solution for $\lim_{x\to\infty}(e^x)^{1/\ln(x)}$

I could have started by simplifying: $(e^a)^b = e^{ab}$, so that

$$\lim_{x\to\infty}(e^x)^{1/\ln(x)} = \lim_{x\to\infty} e^{x/\ln(x)}.$$

Then, since e^x is continuous, we have (using notation $\exp(x) = e^x$)

$$\lim_{x\to\infty} e^{x/\ln(x)} = \exp\left(\lim_{x\to\infty} x/\ln(x)\right)$$

$$\overset{\text{L'H}}{=\!=} \exp\left(\lim_{x\to\infty} 1/(1/x)\right) = \exp\left(\lim_{x\to\infty} x\right) = \infty.$$

Moral: There are no exact rules for how to do these problems. There are just lots of strategies. Get lots of practice!

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $L = \lim_{x\to 0^+} \csc(x) - \cot(x)$.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $L = \lim_{x \to 0^+} \csc(x) - \cot(x)$. Note
   $$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $L = \lim_{x \to 0^+} \csc(x) - \cot(x)$. Note
   $$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{1 - \cos(x)}{\sin(x)}.$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.

   Example: $L = \lim_{x\to 0^+} \csc(x) - \cot(x)$. Note

   $$\csc(x) - \cot(x) = \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \frac{1-\cos(x)}{\sin(x)}.$$

   So by L'Hospital,

   $$L = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{\sin(x)}{\cos(x)} = 0.$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x}) \left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} \right)$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}}$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}}$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

   So

$$L = \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2 - x}}$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

So

$$L = \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2 - x}}\left(\frac{1/x}{1/x}\right)$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

So

$$L = \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2 - x}}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{1 + \sqrt{1 - 1/x}}$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $L = \lim_{x\to\infty} x - \sqrt{x^2 - x}$. Note

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\left(\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}\right)$$

$$= \frac{(x)^2 - (\sqrt{x^2 - x})^2}{x + \sqrt{x^2 - x}} = \frac{x^2 - x^2 + x}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

So

$$L = \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2 - x}}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{1 + \sqrt{1 - 1/x}} = 1/2$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x \to 0^+} \csc(x) - \cot(x) = \lim_{x \to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0$.
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x \to \infty} x - \sqrt{x^2 - x} = \lim_{x \to \infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$.

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

$$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

   $$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2}$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

   $$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$,

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

$$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$, we have

$$e^L = \lim_{x\to\infty} x/e^{x^2} = 0,$$

Solving indeterminate forms of type $\infty - \infty$

This is even less straightforward than exponential forms. Typically, the game is to turn the difference into a fraction. This usually happens one of the following ways:

1. Find a common denominator.
   Example: $\lim_{x\to 0^+} \csc(x) - \cot(x) = \lim_{x\to 0^+} \frac{1-\cos(x)}{\sin(x)} = 0.$
2. Use identities like $(a-b)(a+b) = a^2 - b^2$ to get rid of square roots.
   Example: $\lim_{x\to\infty} x - \sqrt{x^2 - x} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-x}} = 1/2$
3. Take $\exp(L)$ and use $e^{a-b} = e^a/e^b$.
   Example: $L = \lim_{x\to\infty} \ln(x) - x^2$. Start with

   $$e^L = \exp\left(\lim_{x\to\infty} \ln(x) - x^2\right) = \lim_{x\to\infty} e^{\ln(x)-x^2}.$$

   Then since $e^{\ln(x)-x^2} = e^{\ln(x)}/e^{x^2} = x/e^{x^2}$, we have

   $$e^L = \lim_{x\to\infty} x/e^{x^2} = 0, \qquad \text{so } L = -\infty.$$

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \to 0^+} \sin^{-1}(x)/x$
2. $\lim_{x \to 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$
3. $\lim_{x \to \infty} x \sin(\pi/x)$
4. $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$
6. $\lim_{x \to 0} \frac{\tan(x)}{\tanh(x)}$

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.

You try:

For each, find the limit. Use l'Hospital's rule where appropriate. If there is a more elementary method, consider using it.

1. $\lim_{x \to 0^+} \sin^{-1}(x)/x$ — Ans: 1
2. $\lim_{x \to 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$ — Ans: $1/2$
3. $\lim_{x \to \infty} x \sin(\pi/x)$ — Ans: π
4. $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-4x}}{x}$ — Ans: 3
5. $\lim_{x \to \infty} x^{\ln(2)/(1+\ln(x))}$ — Ans: 2
6. $\lim_{x \to 0} \frac{\tan(x)}{\tanh(x)}$ — Ans: 1

Note: For extra practice, go back and prove the claims on the slides with the graphical examples.