Today: 5.6 Hyperbolic functions

Warm up: Let

$$f(x) = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad g(x) = \frac{1}{2}(e^x + e^{-x})$$

Verify the following identities:

(1) $f'(x) = g(x)$ (2) $g'(x) = f(x)$

(3) $f(x)$ is an odd function (i.e. f(-x) = -f(x))

(4) $g(x)$ is an even function (i.e. g(-x) = g(x))

Expand and simplify the following expressions

(1) $g^2(x) - f^2(x)$

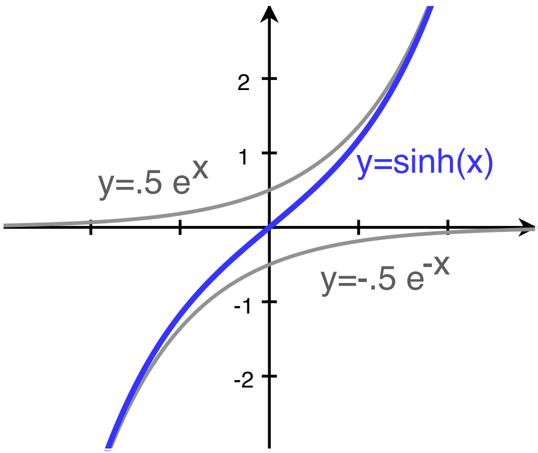
(2) $f(x)g(y) + f(y)g(x)$ (3) $g(x)g(y) + f(x)f(y)$

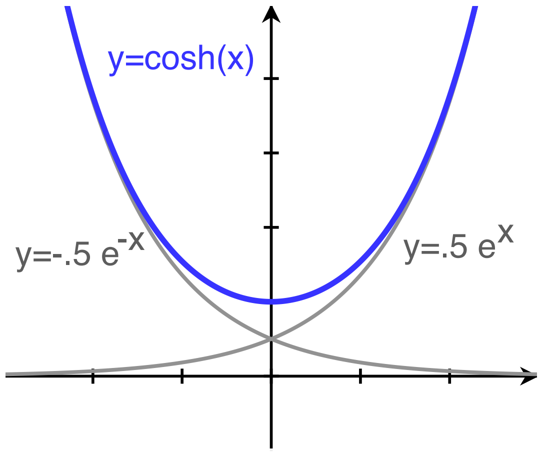
(express the last two as f(stuff) or g(stuff))

Define

“hyperbolic sine” or “sinch” “hyperbolic cosine” or “cosh”

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$





For $x \gg 0$ (large),

$$\sinh(x) \approx \frac{1}{2}e^x \qquad \cosh(x) \approx \frac{1}{2}e^x,$$

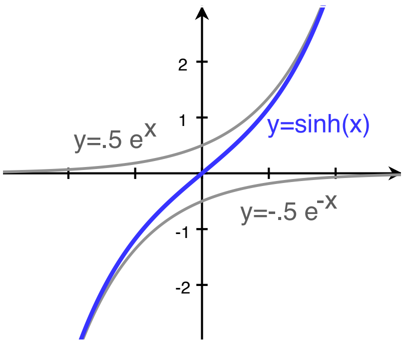
and for $x \ll 0$ (large but negative),

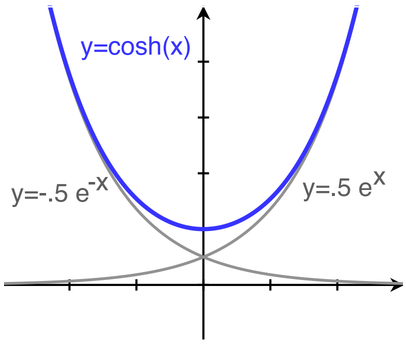
$$\sinh(x) \approx -\frac{1}{2}e^{-x} \qquad \cosh(x) \approx \frac{1}{2}e^{-x}.$$

Define

“hyperbolic sine” or “sinch” “hyperbolic cosine” or “cosh”

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$





We showed:

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\cosh(x)^2 - \sinh^2(x) = 1$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Comparing trig and hyperbolic functions

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$

	Trig	Hyperbolic
even/odd:	$\sin(-x) = -\sin(x)$	$\sinh(-x) = -\sin(x)$
	$\cos(-x) = \cos(x)$	$\cosh(-x) = \cosh(x)$
additive:	$\sin(x+y) = \sin(x)\cos(y)$	$\sinh(x+y) = \sinh(x)\cosh(y)$
	$+\sin(y)\cos(x)$	$+\sinh(y)\cosh(x)$
additive:	$\cos(x+y) = \cos(x)\cos(y)$	$\cosh(x+y) = \cosh(x)\cosh(y)$
	$-\sin(x)\sin(y)$	$+\sinh(x)\sinh(y)$
Pyth.:	$\cos^2(x) + \sin^2(x) = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
ders:	$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\sinh(x) = \cos(x)$
	$\frac{d}{dx}\cos(x) = -\sin(x)$	$\frac{d}{dx}\cosh(x) = +\sinh(x)$

Other hyperbolic functions

For trig functions, we defined

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

$$\sec(x) = 1/\cos(x), \quad \csc(x) = 1/\sin(x), \quad \cot(x) = 1/\tan(x).$$

For hyperbolic functions, similarly define

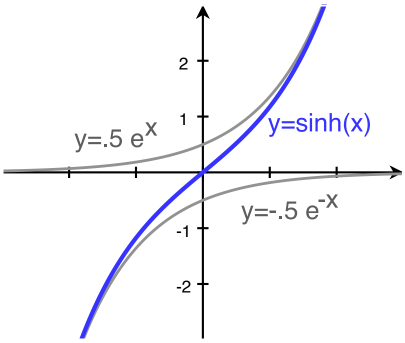
$$\tanh(x) = \sinh(x)/\cosh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

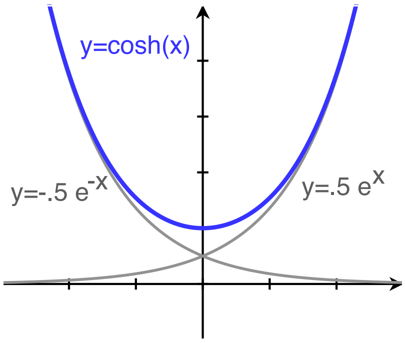
$$\operatorname{sech}(x) = 1/\cosh(x), \quad \operatorname{csch}(x) = 1/\sinh(x), \quad \coth(x) = 1/\tanh(x).$$

("tanch", "sech", "cosech", "cotanch")

Graphs

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$





$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Notice

$$\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}}\right) = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Notice

$$\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}}\right) = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

$$\lim_{x\to-\infty} \tanh(x) = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \overset{(u=-x)}{=\!=\!=} \lim_{u\to\infty} \frac{e^{-u} - e^{u}}{e^{-u} + e^{u}} \left(\frac{e^{-u}}{e^{-u}}\right)$$

$$= \lim_{u\to\infty} \frac{e^{-2u} - 1}{e^{-2u} + 1} = -1$$

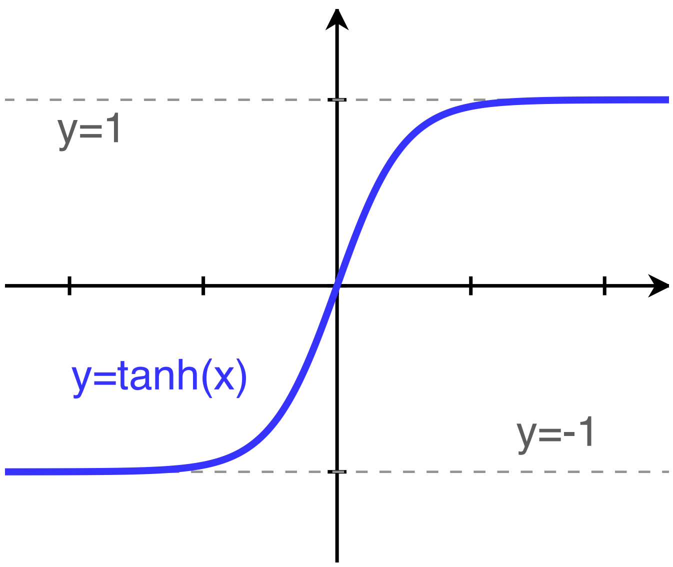
$$\frac{d}{dx}\tanh(x) = \frac{d}{dx}\frac{\sinh(x)}{\cosh(x)} \overset{\text{qu. rule}}{=\!=\!=} \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} > 0$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x\to\infty} \tanh(x) = 1, \qquad \lim_{x\to-\infty} \tanh(x) = -1,$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x) > 0, \qquad \tanh(0) = (1-1)/(1+1) = 0$$



Inverse hyperbolic functions

Define

$$\begin{aligned} \sinh^{-1}(x) = y &\quad\Leftrightarrow\quad \sinh(y) = x \\ \cosh^{-1}(x) = y &\quad\Leftrightarrow\quad \cosh(y) = x \\ \tanh^{-1}(x) = y &\quad\Leftrightarrow\quad \tanh(y) = x \end{aligned}$$

Solving for $y = \sinh^{-1}(x)$:

$$x = \sinh(y) = \tfrac{1}{2}(e^y - e^{-y}), \qquad \text{so that } e^y - 2x - e^{-y} = 0.$$

Thus, multiplying both sides by e^y,

$$0 = e^{2y} - 2xe^y - 1 = u^2 - 2xu - 1, \quad \text{where } u = e^y.$$

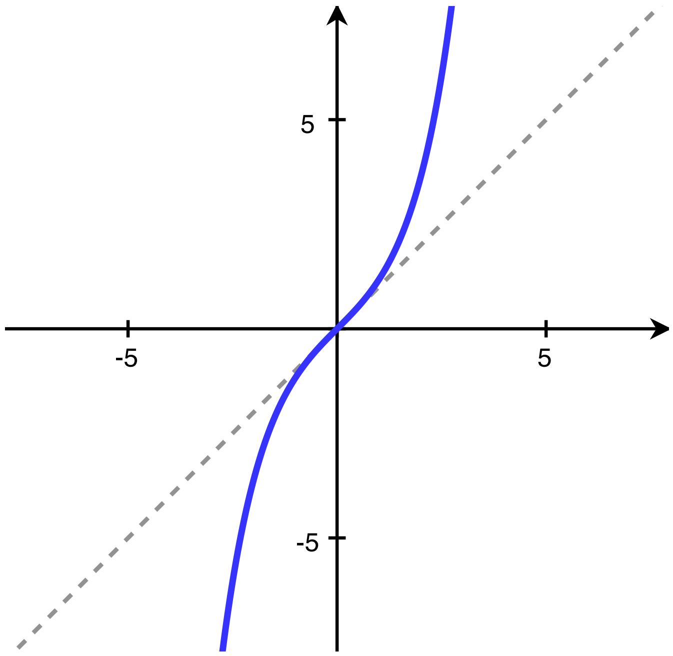
So $e^y = u = (2x \pm \sqrt{4x^2+4})/2 \overset{e^y>0}{=\!=} x + \sqrt{x^2+1}$, so that

$$\boxed{\sinh^{-1}(x) = y = \ln(x + \sqrt{x^2+1}).}$$

You try: Use $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ to calculate a formulas for $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$.

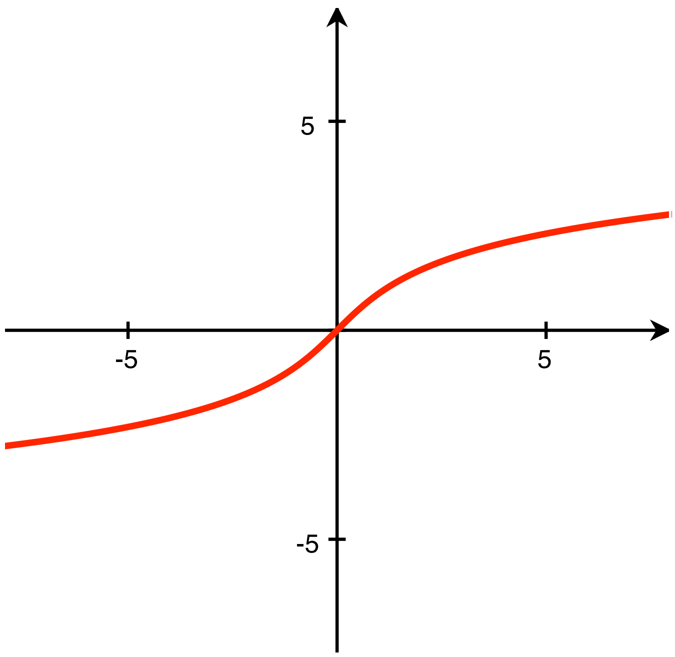
Graphs

$$y = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



Graphs

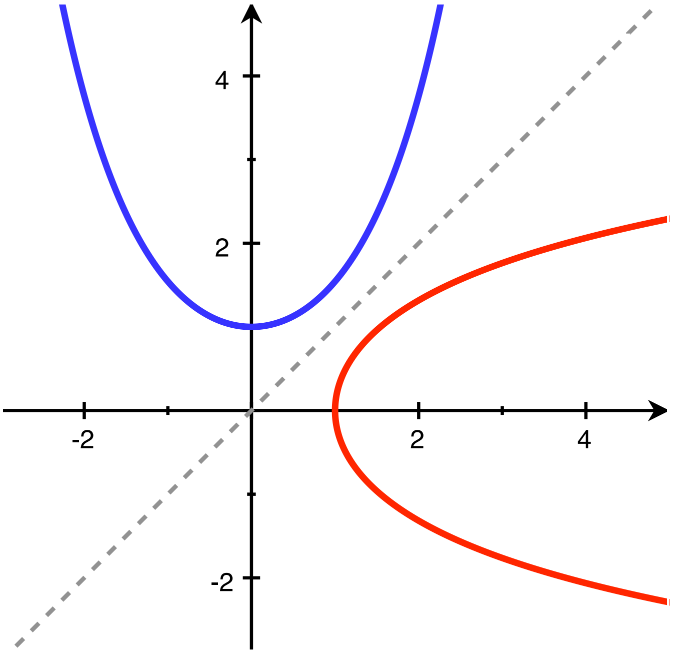
$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$



Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

Graphs

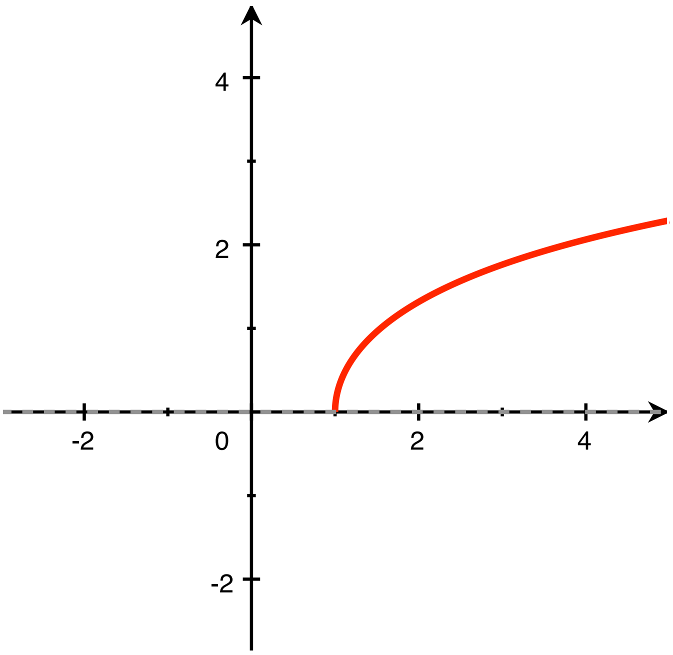
$$y = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$
$$y = \cosh^{-1}(x) = \ln(x \pm \sqrt{x^2 - 1})$$



Domain: $1 \leq x$

Graphs

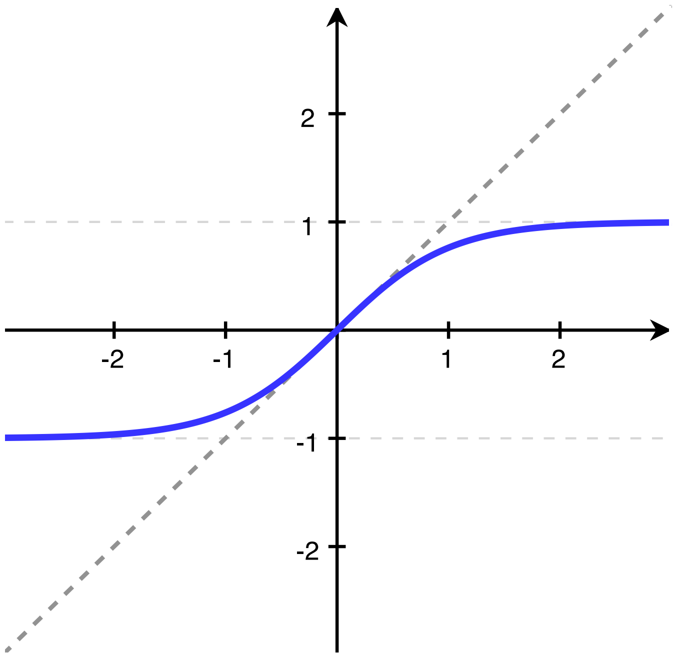
$$y = \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$



Domain: $1 \leq x$ Range: $0 \leq y$

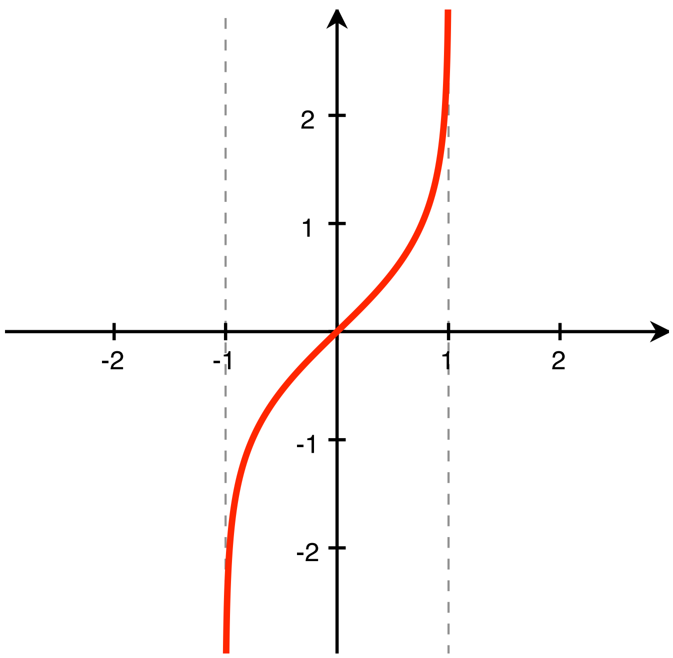
Graphs

$$y = \tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$$



Graphs

$$y = \tanh^{-1}(x) = \frac{1}{2}\ln((1+x)/(1-x))$$



Domain: $(-1, 1)$ Range:$(-\infty, \infty)$

Derivatives and integrals

Using

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \quad \cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1}(x) = \tfrac{1}{2}\ln((1+x)/(1-x)) = \tfrac{1}{2}(\ln(1+x) - \ln(1-x))$$

we have

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1 + x/\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{x+\sqrt{x^2+1}}\right)$$
$$= 1/\sqrt{x^2+1}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1 + x/\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{x+\sqrt{x^2-1}}\right)$$
$$= 1/\sqrt{x^2-1}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \tfrac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \tfrac{1}{2}\left(\frac{1-x+1+x}{(1+x)(1-x)}\right)$$
$$= 1/(1-x^2)$$

In summary:

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \text{etc.}$$

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x),$$
$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}, \quad \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

You try: See integration worksheet at
https://zdaugherty.ccnysites.cuny.edu/teaching/m202s16/

1. $\int \frac{1}{3+x^2}\,dx$
2. $\int \frac{1}{1+3x^2}\,dx$
3. $\int \frac{x}{3+x^2}\,dx$
4. $\int \frac{1}{3-x^2}\,dx$
5. $\int \frac{1}{1-3x^2}\,dx$
6. $\int \frac{x}{3-x^2}\,dx$

7 $\int \frac{1}{\sqrt{3+x^2}}\,dx$

8 $\int \frac{1}{\sqrt{1+3x^2}}\,dx$

9 $\int \frac{x}{\sqrt{3+x^2}}\,dx$

10 $\int \frac{1}{\sqrt{3-x^2}}\,dx$

11 $\int \frac{1}{\sqrt{1-3x^2}}\,dx$

12 $\int \frac{x}{\sqrt{3-x^2}}\,dx$

13 $\int \frac{1}{\sqrt{x^2-3}}\,dx$

14 $\int \frac{1}{\sqrt{3x^2-1}}\,dx$

15 $\int \frac{x}{\sqrt{x^2-3}}\,dx$

16 $\int x\sqrt{x^2-3}\,dx$

17 $\int x\sqrt{3-x^2}\,dx$

18 $\int x\sqrt{1-3x^2}\,dx$

(and 28 more. . .)

5.7: More on limits, indeterminate forms, and L'Hospital's rule

Consider the function

$$F(x) = \frac{\ln(x)}{x-1}.$$

As $x \to 1$, both the numerator and the denominator approach 0. Both approach somewhat slowly, but does one go faster than the other? Or does it approach some interesting ratio? Similar question for $x \to \infty$, where both the numerator and denominator approach ∞.
Indeterminate forms are ratios where the numerator and the denominator each either approach 0, or each approach $\pm\infty$.
So far, we've been able to calculate limits with indeterminate forms through algebraic tricks or substitution, or recognizing limits as derivatives.

Past examples of solving indeterminate forms

1. $\lim_{x\to\infty} \frac{3x^2+x}{5x^2-1}\left(\frac{x^{-2}}{x^{-2}}\right) = \lim_{x\to\infty} \frac{3+x^{-1}}{5-x^{-2}} = \frac{3}{5}$
2. $\lim_{x\to-\infty} \frac{3e^{2x}+e^{x}}{5e^{2x}-e^{x}}\left(\frac{e^{-x}}{e^{-x}}\right) = \lim_{x\to\infty} \frac{3e^{x}+1}{5e^{x}-1} = \frac{0+1}{0-1} = -1$
3. $\lim_{x\to\pi} \frac{e^{\sin(x)}-1}{x-\pi}$

   Recall, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

   Note, $e^{\sin(x)}\Big|_{x=\pi} = e^{\sin(\pi)} = e^0 = 1$.

   So

$$\lim_{x\to\pi} \frac{e^{\sin(x)}-1}{x-\pi} = \frac{d}{dx}e^{\sin(x)}\Big|_{x=\pi} = \cos(x)e^{\sin(x)}\Big|_{x=\pi} = (-1)e^0 = -1.$$

So similarly, since $\ln(1) = 0$,

$$\lim_{x\to1} \frac{\ln(x)}{x-1} = \frac{d}{dx}\ln(x)\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1.$$

L'Hospital's rule

L'Hospital's rule relates the limit of the ratio of two functions to the limit of the ratio of their derivatives.
Consider differentiable functions $f(x)$ and $g(x)$ such that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x),$$

and $g'(x) \neq 0$ for x close to but not equal to a. Then

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{f(x)}{x-a}, \text{ and}$$

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{g(x)}{x-a}.$$

(If f or g are not defined at a, we can work around this: see appendix C in book)
So

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f(x)(x-a)}{(x-a)g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

L'Hospital's rule

Theorem

Suppose f and g are differentiable functions and $g'(x) \neq 0$ for x close to but not equal to a. Suppose that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \quad \text{or} \quad \lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x).$$

Then if the limit of $f'(x)/g'(x)$ as $x \to a$ exists (or is $\pm\infty$), we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same holds for $x \to \pm\infty$ and one-sided limits $x \to a^{\pm}$.

Example. Let's recheck $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.
$\ln(x)$ and $x-1$ differentiable? ✓ $g'(x) = 1 \neq 0$ ✓,
$\ln(x) \to 0$ and $x-1 \to 0$ as $x \to 1$ ✓

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1 ✓$$

You try

For each of the following, verify that you can use L'Hospital's rule to calculate the limit, and then do so.

$$(1)\ \lim_{x\to\pi} \frac{e^{\sin(x)}-1}{x-\pi} \quad (2)\ \lim_{x\to\infty} \frac{e^x}{x} \quad (3)\ \lim_{x\to\infty} \frac{\ln(x)}{x}$$

Each of the following has some reason why you can't use L'Hospital's rule. For each, what is the reason?

$$(1)\ \lim_{x\to 0} \frac{x}{|x|} \quad (2)\ \lim_{x\to 0^+} \frac{x}{\lfloor x \rfloor} \quad (3)\ \lim_{x\to\pi} \frac{\sin(x)}{1-\cos(x)}$$

(Recall, $\lfloor x \rfloor$ is the *floor* function, and gives back the biggest integer less than or equal to x, i.e. $\lfloor 2.1 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$, $\lfloor 1 \rfloor = 1$, etc..)