

Today: 5.6 Hyperbolic functions

Warm up: Let

$$f(x) = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad g(x) = \frac{1}{2}(e^x + e^{-x})$$

Verify the following identities:

$$(1) f'(x) = g(x) \quad (2) g'(x) = f(x)$$

$$(3) f(x) \text{ is an odd function (i.e. } f(-x) = -f(x))$$

$$(4) g(x) \text{ is an even function (i.e. } g(-x) = g(x))$$

Expand and simplify the following expressions

$$(1) g^2(x) - f^2(x)$$

$$(2) f(x)g(y) + f(y)g(x) \quad (3) g(x)g(y) + f(x)f(y)$$

(express the last two as $f(\text{stuff})$ or $g(\text{stuff})$)

Define

“hyperbolic sine” or “sinch”

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

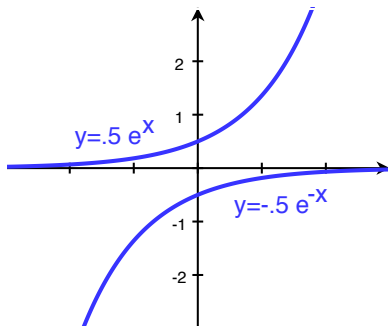
“hyperbolic cosine” or “cosh”

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

Define

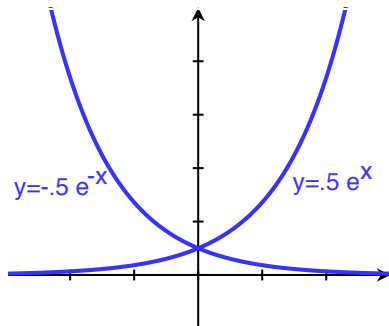
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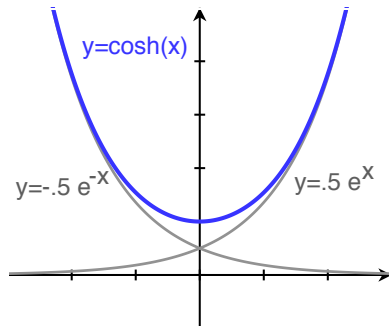
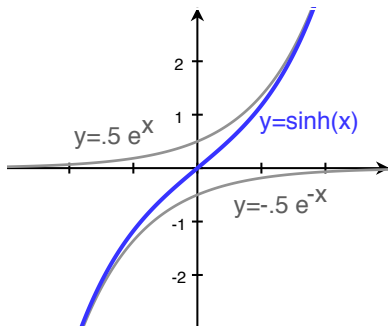
Define

“hyperbolic sine” or “sinch”

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

“hyperbolic cosine” or “cosh”

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



For $x \gg 0$ (large),

$$\sinh(x) \approx \frac{1}{2}e^x \quad \cosh(x) \approx \frac{1}{2}e^x,$$

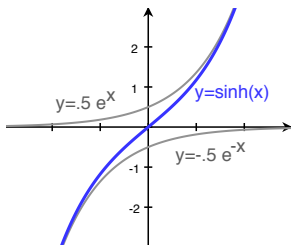
and for $x \ll 0$ (large but negative),

$$\sinh(x) \approx -\frac{1}{2}e^{-x} \quad \cosh(x) \approx \frac{1}{2}e^{-x}.$$

Define

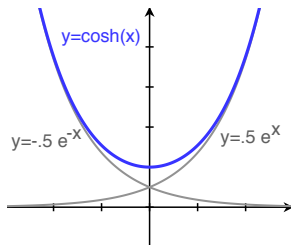
“hyperbolic sine” or “sinch”

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



“hyperbolic cosine” or “cosh”

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



We showed:

$$\sinh(-x) = -\sinh(x)$$

$$\cosh(-x) = \cosh(x)$$

$$\frac{d}{dx} \sinh(x) = \cosh(x)$$

$$\frac{d}{dx} \cosh(x) = \sinh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sinh(x + y) = \sinh(x) \cosh(y) + \sinh(y) \cosh(x)$$

$$\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y)$$

Comparing trig and hyperbolic functions

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

	Trig	Hyperbolic
even/odd:	$\sin(-x) = -\sin(x)$ $\cos(-x) = \cos(x)$	$\sinh(-x) = -\sinh(x)$ $\cosh(-x) = \cosh(x)$
additive:	$\sin(x + y) = \sin(x)\cos(y)$ $+ \sin(y)\cos(x)$	$\sinh(x + y) = \sinh(x)\cosh(y)$ $+ \sinh(y)\cosh(x)$
additive:	$\cos(x + y) = \cos(x)\cos(y)$ $- \sin(x)\sin(y)$	$\cosh(x + y) = \cosh(x)\cosh(y)$ $+ \sinh(x)\sinh(y)$
Pyth.:	$\cos^2(x) + \sin^2(x) = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
ders:	$\frac{d}{dx} \sin(x) = \cos(x)$ $\frac{d}{dx} \cos(x) = -\sin(x)$	$\frac{d}{dx} \sinh(x) = \cosh(x)$ $\frac{d}{dx} \cosh(x) = \sinh(x)$

Comparing trig and hyperbolic functions

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

	Trig	Hyperbolic
even/odd:	$\sin(-x) = -\sin(x)$ $\cos(-x) = \cos(x)$	$\sinh(-x) = -\sinh(x)$ $\cosh(-x) = \cosh(x)$
additive:	$\sin(x + y) = \sin(x)\cos(y)$ $+ \sin(y)\cos(x)$	$\sinh(x + y) = \sinh(x)\cosh(y)$ $+ \sinh(y)\cosh(x)$
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Other hyperbolic functions

For trig functions, we defined

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

$$\sec(x) = 1/\cos(x), \quad \csc(x) = 1/\sin(x), \quad \cot(x) = 1/\tan(x).$$

Other hyperbolic functions

For trig functions, we defined

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

$$\sec(x) = 1/\cos(x), \quad \csc(x) = 1/\sin(x), \quad \cot(x) = 1/\tan(x).$$

For hyperbolic functions, similarly define

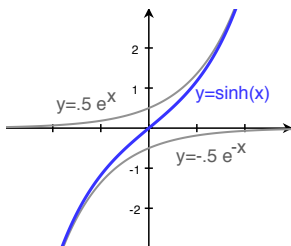
$$\tanh(x) = \sinh(x)/\cosh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech}(x) = 1/\cosh(x), \quad \operatorname{csch}(x) = 1/\sinh(x), \quad \operatorname{coth}(x) = 1/\tanh(x).$$

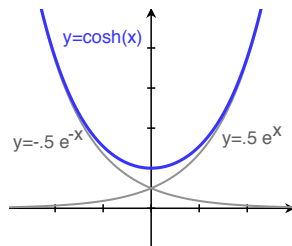
(“tanch”, “sech”, “cosech”, “cotanch”)

Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

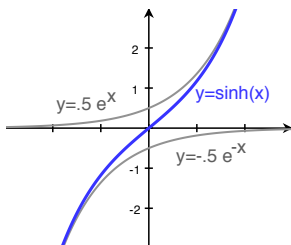


$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

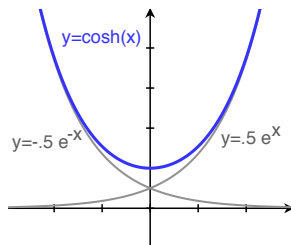


Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



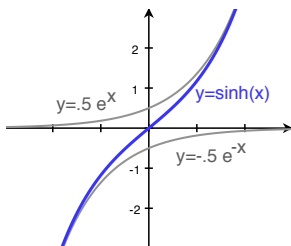
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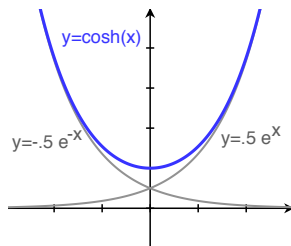
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



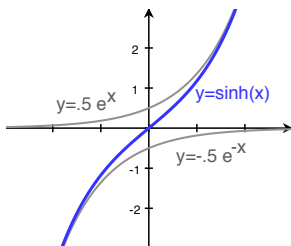
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Notice

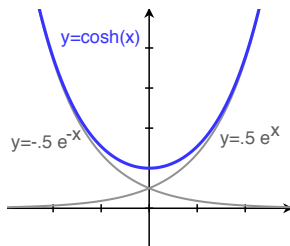
$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



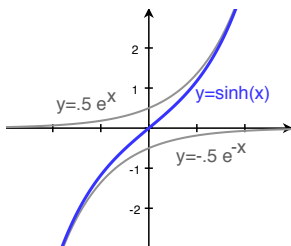
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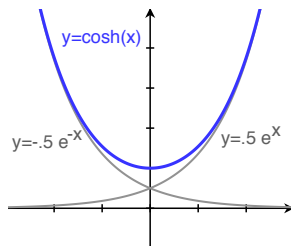
$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right)$$

Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



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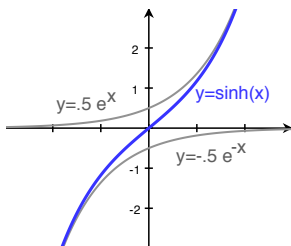
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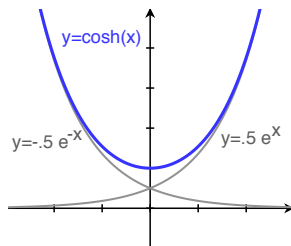
$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Graphs

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$



$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Notice

$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

Graphs

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$$\lim_{x \rightarrow -\infty} \tanh(x)$$

Graphs

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Graphs

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Graphs

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Notice

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$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

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$$\frac{d}{dx} \tanh(x)$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Notice

$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

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$$\frac{d}{dx} \tanh(x) = \frac{d}{dx} \frac{\sinh(x)}{\cosh(x)}$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

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$$\frac{d}{dx} \tanh(x) = \frac{d}{dx} \frac{\sinh(x)}{\cosh(x)} \stackrel{\text{qu. rule}}{=} \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

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$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

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Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Notice

$$\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \left(\frac{e^{-x}}{e^{-x}} \right) = \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

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Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x \rightarrow \infty} \tanh(x) = 1, \quad \lim_{x \rightarrow -\infty} \tanh(x) = -1,$$

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x) > 0,$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

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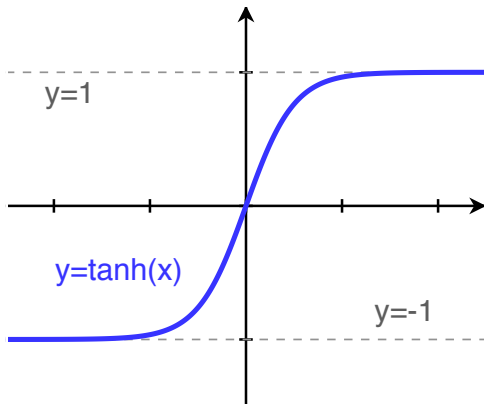
$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x) > 0, \quad \tanh(0) = (1 - 1)/(1 + 1) = 0$$

Graphs

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x \rightarrow \infty} \tanh(x) = 1, \quad \lim_{x \rightarrow -\infty} \tanh(x) = -1,$$

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Inverse hyperbolic functions

Define

$$\sinh^{-1}(x) = y \quad \Leftrightarrow \quad \sinh(y) = x$$

$$\cosh^{-1}(x) = y \quad \Leftrightarrow \quad \cosh(y) = x$$

$$\tanh^{-1}(x) = y \quad \Leftrightarrow \quad \tanh(y) = x$$

Inverse hyperbolic functions

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Solving for $y = \sinh^{-1}(x)$:

Inverse hyperbolic functions

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Solving for $y = \sinh^{-1}(x)$:

$$x = \sinh(y)$$

Inverse hyperbolic functions

Define

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Solving for $y = \sinh^{-1}(x)$:

$$x = \sinh(y) = \frac{1}{2}(e^y - e^{-y}),$$

Inverse hyperbolic functions

Define

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$$x = \sinh(y) = \frac{1}{2}(e^y - e^{-y}), \quad \text{so that } e^y - 2x - e^{-y} = 0.$$

Inverse hyperbolic functions

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So $e^y = u$

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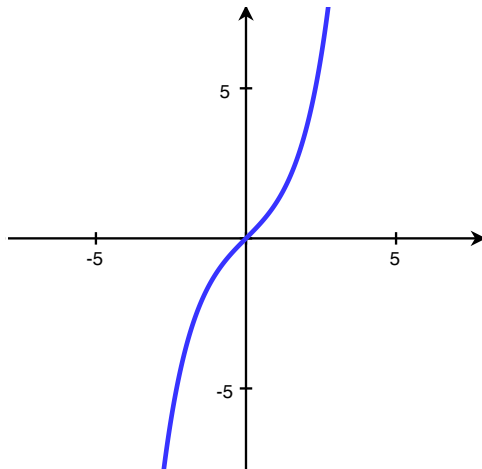
So $e^y = u = (2x \pm \sqrt{4x^2 + 4})/2 \stackrel{e^y > 0}{=} x + \sqrt{x^2 + 1}$, so that

$$\boxed{\sinh^{-1}(x) = y = \ln(x + \sqrt{x^2 + 1}).}$$

You try: Use $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ to calculate a formulas for $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$.

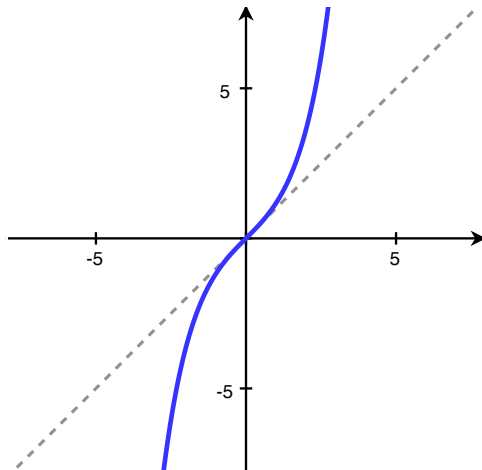
Graphs

$$y = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



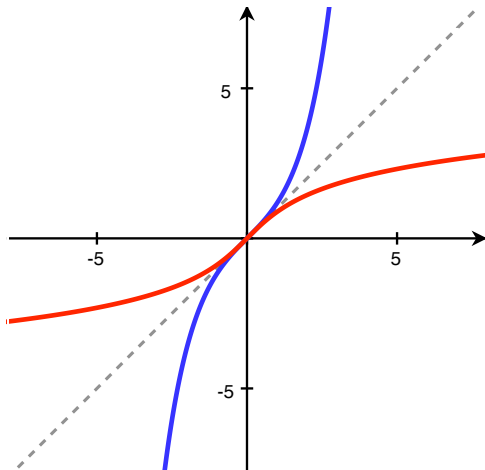
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Graphs

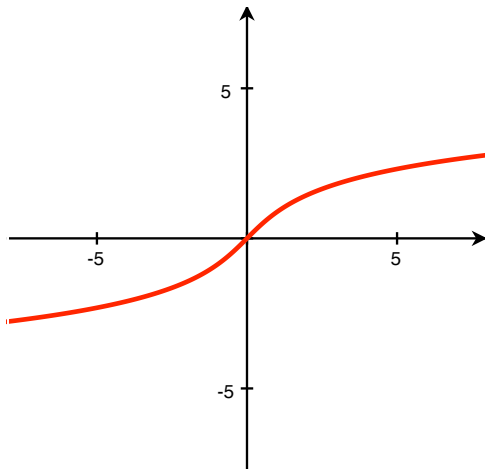
$$y = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$
$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$



Domain: $(-\infty, \infty)$

Graphs

$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

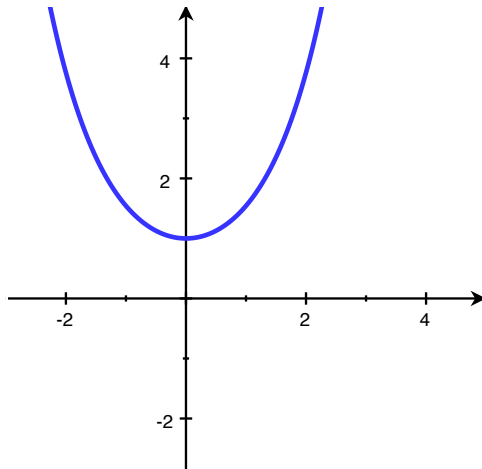


Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

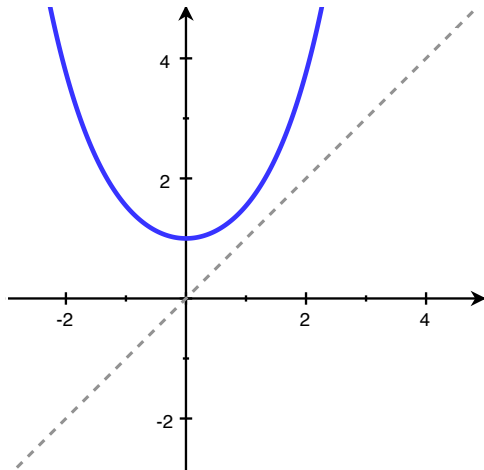
Graphs

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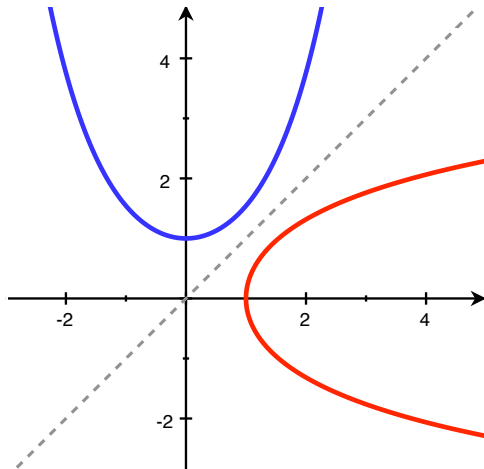
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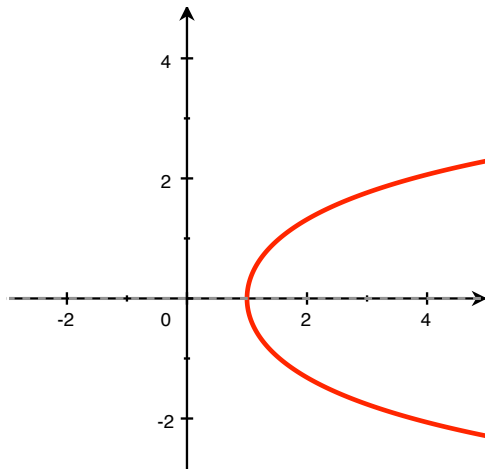
$$y = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$
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Domain: $1 \leq x$

Graphs

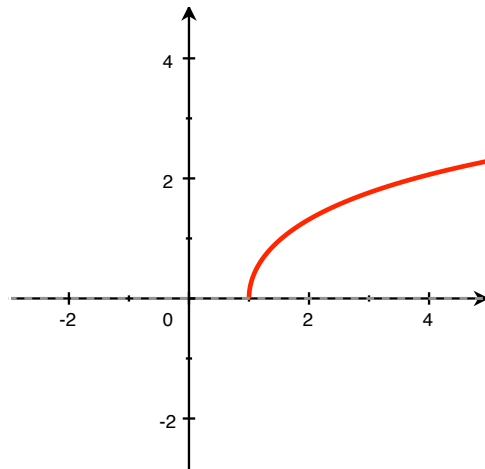
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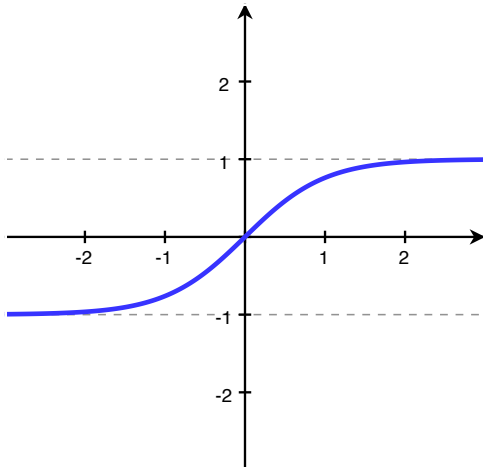


Domain: $1 \leq x$

Range: $0 \leq y$

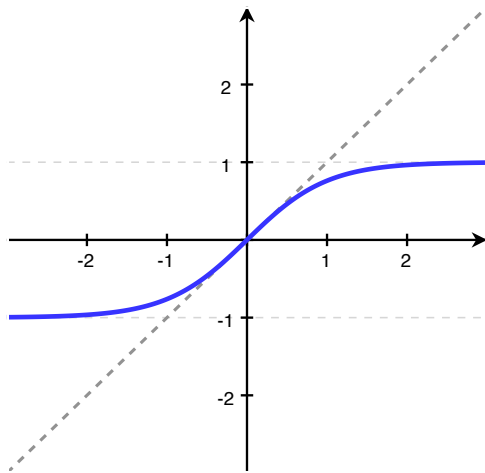
Graphs

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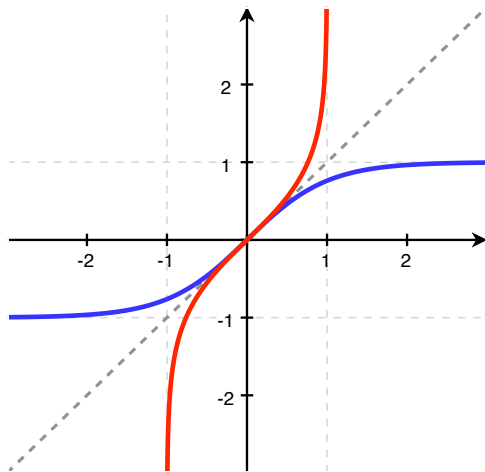
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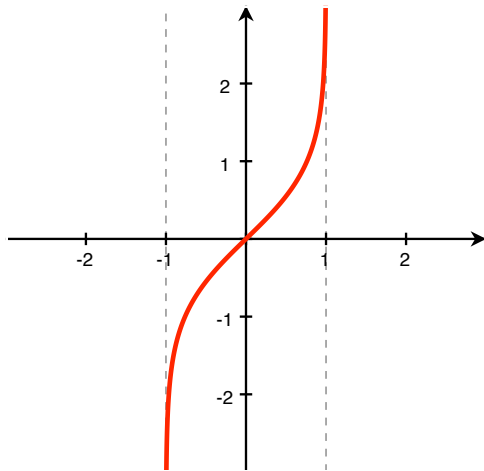
$$y = \tanh(x) = (e^x - e^{-x}) / (e^x + e^{-x})$$
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Domain: $(-1, 1)$

Graphs

$$y = \tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$



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Derivatives and integrals

Using

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}), \quad \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$

$$\tanh^{-1}(x) = \frac{1}{2} \ln((1+x)/(1-x)) = \frac{1}{2}(\ln(1+x) - \ln(1-x))$$

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Derivatives and integrals

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In summary:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \text{etc.}$$

$$\frac{d}{dx} \sinh(x) = \cosh(x), \quad \frac{d}{dx} \cosh(x) = \sinh(x),$$

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$$

$$\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2 + 1}}, \quad \frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}},$$

$$\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1 - x^2}$$

You try: See integration worksheet at

<https://zdaugherty.ccnysites.cuny.edu/teaching/m202s16/>

1. $\int \frac{1}{3+x^2} dx$

7. $\int \frac{1}{\sqrt{3+x^2}} dx$

13. $\int \frac{1}{\sqrt{x^2-3}} dx$

2. $\int \frac{1}{1+3x^2} dx$

8. $\int \frac{1}{\sqrt{1+3x^2}} dx$

14. $\int \frac{1}{\sqrt{3x^2-1}} dx$

3. $\int \frac{x}{3+x^2} dx$

9. $\int \frac{x}{\sqrt{3+x^2}} dx$

15. $\int \frac{x}{\sqrt{x^2-3}} dx$

4. $\int \frac{1}{3-x^2} dx$

10. $\int \frac{1}{\sqrt{3-x^2}} dx$

16. $\int x\sqrt{x^2-3} dx$

5. $\int \frac{1}{1-3x^2} dx$

11. $\int \frac{1}{\sqrt{1-3x^2}} dx$

17. $\int x\sqrt{3-x^2} dx$

6. $\int \frac{x}{3-x^2} dx$

12. $\int \frac{x}{\sqrt{3-x^2}} dx$

18. $\int x\sqrt{1-3x^2} dx$

(and 28 more...)

5.7: More on limits, indeterminate forms, and L'Hospital's rule

Consider the function

$$F(x) = \frac{\ln(x)}{x-1}.$$

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As $x \rightarrow 1$, both the numerator and the denominator approach 0. Both approach somewhat slowly, but does one go faster than the other? Or does it approach some interesting ratio?

5.7: More on limits, indeterminate forms, and L'Hospital's rule

Consider the function

$$F(x) = \frac{\ln(x)}{x - 1}.$$

As $x \rightarrow 1$, both the numerator and the denominator approach 0. Both approach somewhat slowly, but does one go faster than the other? Or does it approach some interesting ratio? Similar question for $x \rightarrow \infty$, where both the numerator and denominator approach ∞ .

Indeterminate forms are ratios where the numerator and the denominator each either approach 0, or each approach $\pm\infty$. So far, we've been able to calculate limits with indeterminate forms through algebraic tricks or substitution, or recognizing limits as derivatives.

Past examples of solving indeterminate forms

1. $\lim_{x \rightarrow \infty} \frac{3x^2 + x}{5x^2 - 1}$

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Past examples of solving indeterminate forms

$$1. \lim_{x \rightarrow \infty} \frac{3x^2 + x}{5x^2 - 1} \left(\frac{x^{-2}}{x^{-2}} \right) = \lim_{x \rightarrow \infty} \frac{3 + x^{-1}}{5 - x^{-2}}$$

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$$1. \lim_{x \rightarrow \infty} \frac{3x^2 + x}{5x^2 - 1} \left(\frac{x^{-2}}{x^{-2}} \right) = \lim_{x \rightarrow \infty} \frac{3 + x^{-1}}{5 - x^{-2}} = \frac{3}{5}$$

Past examples of solving indeterminate forms

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$$2. \lim_{x \rightarrow -\infty} \frac{3e^{2x} + e^x}{5e^{2x} - e^x}$$

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But what about

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L'Hospital's rule relates the limit of the ratio of two functions to the limit of the ratio of their derivatives.

Consider differentiable functions $f(x)$ and $g(x)$ such that

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You try

For each of the following, verify that you can use L'Hospital's rule to calculate the limit, and then do so.

$$(1) \lim_{x \rightarrow \pi} \frac{e^{\sin(x)} - 1}{x - \pi} \quad (2) \lim_{x \rightarrow \infty} \frac{e^x}{x} \quad (3) \lim_{x \rightarrow \infty} \frac{\ln(x)}{x}$$

Each of the following has some reason why you can't use L'Hospital's rule. For each, what is the reason?

$$(1) \lim_{x \rightarrow 0} \frac{x}{|x|} \quad (2) \lim_{x \rightarrow 0^+} \frac{x}{\lfloor x \rfloor} \quad (3) \lim_{x \rightarrow \pi} \frac{\sin(x)}{1 - \cos(x)}$$

(Recall, $\lfloor x \rfloor$ is the *floor* function, and gives back the biggest integer less than or equal to x , i.e. $\lfloor 2.1 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$, $\lfloor 1 \rfloor = 1$, etc..)

