Today: 5.4 General log and exp functions (continued)

Warm up:

$$\log_a(x) = \ln(x)/\ln(a) \qquad \frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$$

1. Evaluate the following functions.

$$\log_5(25) \qquad \log_7\sqrt{7} \qquad \log_4 8 - \log_4 2$$

2. Differentiate the following functions.

$$\log_{10} x \qquad x\log_2(x) \qquad 2^{x+\log_3(x)}$$

$$\log_5(x^2+1), \qquad x\log_e(x) - x, \qquad 3^{x\ln(x)}, \qquad \sqrt{1-(1/3)^x}$$

3. Calculate the following antiderivatives:

$$\int \frac{3^x}{3^x+3}\,dx \qquad \int \frac{2^{1/x}}{x^2}\,dx$$

$$\int e^x(3e^x+1)^{1/3}\,dx \qquad \int \frac{1}{x\ln(x)}\,dx$$

You try:

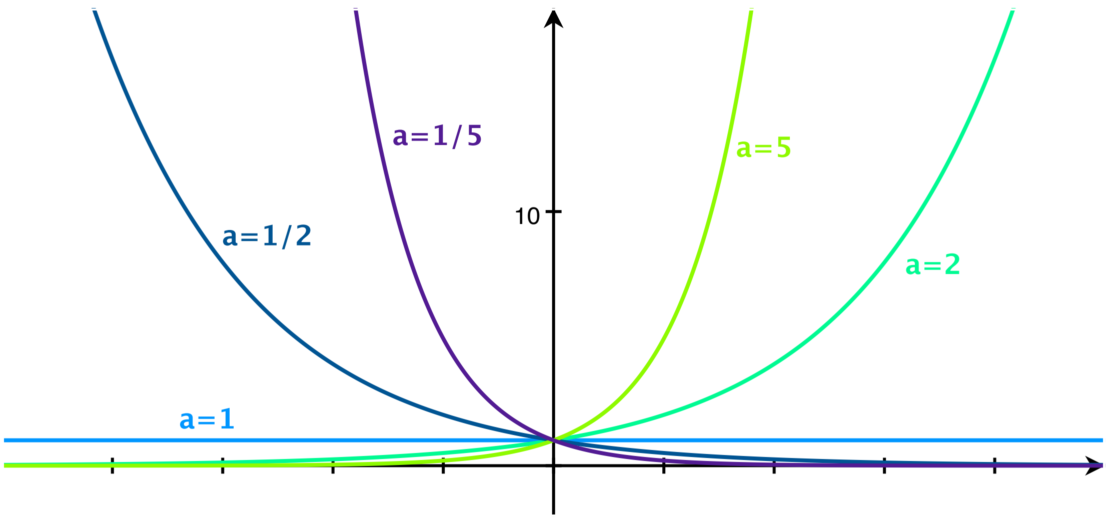
Recall $\log_a(x) = \ln(x)/\ln(a)$, so that $\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$.

Complete a similar analysis of the graphs of $y = \log_a(x)$. Namely,

- Break into cases based on a (there should be three, one of which is very strange).
- Analyze the increasing/decreasing behavior in each of the cases.
- Describe the graph transformations that take you from the graph of $\ln(x)$ to $\log_a(x)$ in each of the cases.
- Graph $y = \log_a(x)$ for a few representative values of a on the same axis (like we did on the last slide for $y = a^x$), including $a = e$ and $a = 1/e$. (Recall $2 < e < 3$.)

Check your reasoning against the fact that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other across the line $y = x$.

Graphs

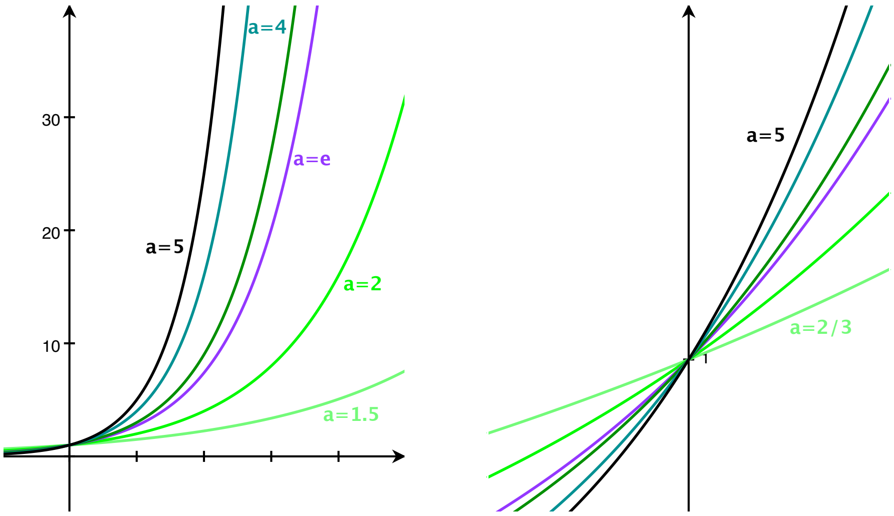


$$y = a^x, \qquad \frac{dy}{dx} = \ln(a)a^x.$$

$0 < a < 1$: Slope always negative, monotonically decreasing function.
$a = 1$: Slope always zero, constant function.
$1 < a$: Slope always positive, monotonically increasing function.

Graphs

$y = a^x, \qquad a > 1$



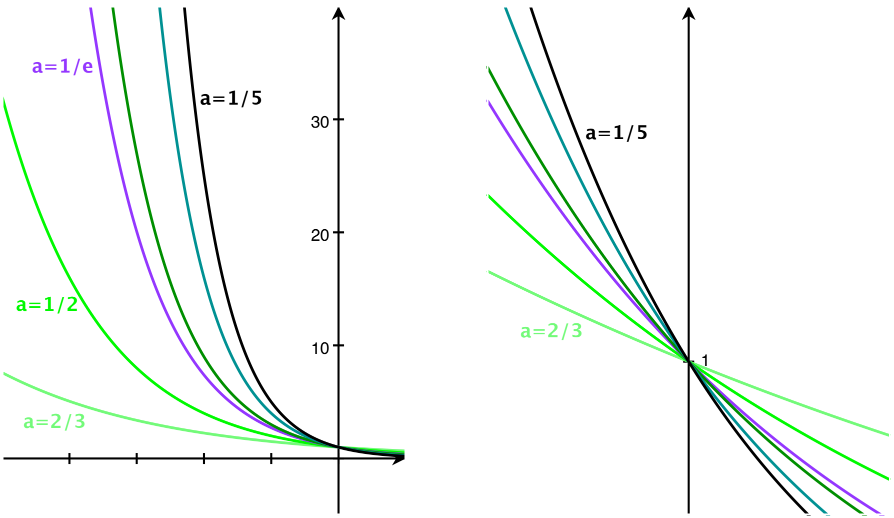
Notice: Slope at $x = 0$ is

$$\frac{d}{dx} a^x \Big|_{x=0} = \ln(a) a^x \Big|_{x=0} = \ln(a).$$

So $y = e^x$ is the exponential function whose slope through the point $(0, 1)$ is 1. For $a < e$, the slope at $x = 0$ is less than 1, and for $a > e$, the slope at $x = 0$ is greater than 1.

Graphs

$y = a^x, \qquad a < 1$



Notice:

$(1/e)^x = e^{-x}, \qquad$ Graph transformation: flip over y-axis, then stretch.

So $y = (1/e)^x$ is the exponential function whose slope through the point $(0, 1)$ is -1. For $a < 1/e$, the slope at $x = 0$ is less than -1, and for $a > 1/e$, the slope at $x = 0$ is greater than -1.

You try:

Recall $\log_a(x) = \ln(x)/\ln(a)$, so that $\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$.

Complete a similar analysis of the graphs of $y = \log_a(x)$. Namely,

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Continuously compounded interest and e as a limit

Say you put $1,000 into a savings account with a 1% interest rate. How much will you have at the end of the year?
If the interest is calculated once, at the end of the year, then you would have

$$\$1,000 * (1 + .01) = \$1,010.$$

Suppose, instead, that 1% is spread out over the course of the year, and compounded monthly. Then you'll have

$$\begin{aligned} \$1,000 * (1 + .01(\tfrac{1}{12})) &= \$1,000.833 && \text{at the end of month 1,} \\ \$1,000 * (1 + .01(\tfrac{1}{12}))^2 &= \$1,001.667 && \text{at the end of month 2,} \\ \$1,000 * (1 + .01(\tfrac{1}{12}))^3 &= \$1,002.502 && \text{at the end of month 3,} \\ &\vdots \\ \$1,000 * (1 + .01(\tfrac{1}{12}))^{12} &= \$1,010.046 && \text{at the end of month 12.,} \end{aligned}$$

i.e. a whole 4.6 cents more than if it had been compounded at the end of the year.

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$$\$1,000 * (1 + .01(\tfrac{1}{12}))^{12} = \$1,010.046 \text{ at the end of the year.}$$

Similarly, if that 1% is compounded weekly, you'll have

$$\$1,000 * (1 + .01(\tfrac{1}{52}))^{52} = \$1,010.049 \text{ at the end of the year,}$$

or compounded daily, you'll have

$$\$1,000 * (1 + .01(\tfrac{1}{365}))^{365} = \$1,010.050 \text{ at the end of the year,}$$

and so on.

Continuously compounded interest and e as a limit

Say you put \$1,000 into a savings account with a 1% interest rate. How much will you have at the end of the year?
If the interest is compounded n times a year, then you'll have

$$\$1,000 * (1 + .01(\tfrac{1}{n}))^n \text{ at the end of the year.}$$

In general, if the deposited amount is D, the interest rate is r, and the number of times it's compounded is n, you end up with

$$D(1 + r(\tfrac{1}{n}))^n \text{ at the end of the year.}$$

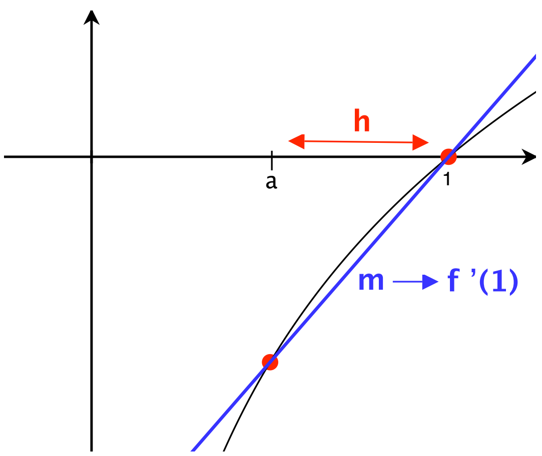
Continuously compounded interest is when you let $n \to \infty$:

$$\begin{aligned}\lim_{n\to\infty} D(1 + r(\tfrac{1}{n}))^n &= D \lim_{n\to\infty} (1 + (\tfrac{r}{n}))^n \\ &= D \lim_{n\to\infty} \left((1 + (\tfrac{r}{n}))^{n/r}\right)^r \\ &= D \left(\lim_{n\to\infty} (1 + (\tfrac{r}{n}))^{n/r}\right)^r.\end{aligned}$$

Continuously compounded interest and e as a limit

Continuously compounded interest: $D\left(\lim_{n\to\infty} (1 + (\tfrac{r}{n}))^{n/r}\right)^r$

Back to logarithms: Let $f(x) = \ln(x)$.Then $f'(x) = 1/x$ and $f'(1) = 1$. But let's think back to what derivatives really are:



$$\begin{aligned}f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} \\ &= \lim_{h\to 0} \tfrac{1}{h} \ln(1+h) \\ &= \lim_{h\to 0} \ln\left((1+h)^{1/h}\right) \\ &= \ln\left(\lim_{h\to 0} (1+h)^{1/h}\right)\end{aligned}$$

Since $1 = \ln\left(\lim_{h\to 0}(1+h)^{1/h}\right)$, we have $\boxed{e = \lim_{h\to 0}(1+h)^{1/h}}$.

If I deposit D into an account that accrues interest at a rate r, compounded continuously, I will have

$$D\left(\lim_{n\to\infty}(1+(\tfrac{r}{n}))^{n/r}\right)^r \qquad \text{at the end of one year,}$$

and

$$D\left(\left(\lim_{n\to\infty}(1+(\tfrac{r}{n}))^{n/r}\right)^r\right)^t = \left(\lim_{n\to\infty}(1+(\tfrac{r}{n}))^{n/r}\right)^{tr}$$

at the end of t years. On the other hand,

$$e = \lim_{h\to 0}(1+h)^{1/h}.$$

Putting these together, notice that

$$\text{as } n\to\infty, \qquad \text{we have } r/n \to 0.$$

So letting $h = r/n$,

$$\lim_{n\to\infty}(1+(\tfrac{r}{n}))^{n/r} = \lim_{h\to 0}(1+h)^{1/h} = e.$$

So we'll have

$$De^{rt} \qquad \text{at the end of } t \text{ years.}$$

5.5 Exponential growth and decay

Say a population, at least initially, grows at a rate proportional to its size (think bacteria in a petrie dish).

$$y = \text{pop. size}\ , \qquad \frac{dy}{dt} = ky, \qquad \text{where } k > 0 \text{ is fixed.}$$

Or say a mass of radioactive substance decays at a rate proportional to its size.

$$y = \text{mass}\ , \qquad \frac{dy}{dt} = ky, \qquad \text{where } k < 0 \text{ is fixed.}$$

The equation $\frac{dy}{dt} = ky$ is called a differential equation, because it is an equation describing the function in terms of its derivative. This particular differential equation is sometimes called the law of natural growth (if $k > 0$) or natural decay (if $k < 0$). We call k the proportionality constant.

5.5 Exponential growth and decay

$$\boxed{\frac{dy}{dt} = ky, \qquad \text{where } k \text{ is fixed.}}$$

A solution to a differential equation is a function $y = f(t)$ that satisfies this equation.
What is a function whose derivative is itself times a constant?

$$y = e^{kt} \qquad \text{is one solution!}$$

So are $2e^{kt}$, $3e^{kt}$, $-e^{kt}\ldots$ Every solution looks like

$$\boxed{y = Ce^{kt}} \qquad \text{where } C \text{ and } k \text{ are constant.}$$

We call this the general solution. Notice that

$$y(0) = Ce^0 = C.$$

Example 0: Compound interest

Let y be the amount of money we have in the bank at time t. We saw before that if we deposit $D = y(0)$ into an account that accrues interest at a rate r, compounded continuously, then we'll have

$$y = y(0)e^{rt} \qquad \text{at the end of } t \text{ years.}$$

Well this is equivalent to the fact that our money is growing continuously at a rate

$$\frac{dy}{dt} = r\left(y(0)e^{rt}\right) = ry,$$

i.e. it's growing proportionally to its size.So continuously compounded interest was a natural growth problem.

Example 1: Population growth (see ex 1 in §5.5)

In 1950, the world population was about 2560 million. In 1960, it was about 3040 million. Assuming population grows according to the natural growth model, estimate the population in (1)1993, and (2) 2020. The ingredients:

$$\frac{dy}{dt} = ky, \qquad \text{so that } y = y(0)e^{kt}.$$

$$\begin{aligned} y &= \text{population (in millions)} \\ t &= \#\text{ years since 1950} \\ y(0) &= 2560 \\ k &= \text{ growth rate (which we need to calculate):} \end{aligned}$$

Use the other data point:

$$3040 = y(10) = 2560e^{k(10)} \quad \text{so } k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right) \approx .0172.$$

So (1) $y(1993 - 1950) = y(43) \approx 2560e^{(.0172)43} \approx 5360$ million, and (2) $y(2020 - 1950) = y(70) \approx 2560e^{(.0172)70} \approx 8520$ million.

Example 2: Radioactive decay (see ex 2 in §5.5)

The half-life of a radioactive substance is the amount of time that it takes for the substance to decay to half its original amount. In other words, if

$$y = \text{mass of substance at time } t, \quad m_0 = y(0)$$

and $\quad T = \text{half-life of substance,} \qquad$ then $\quad y(T) = \frac{1}{2}m_0.$ $(*)$

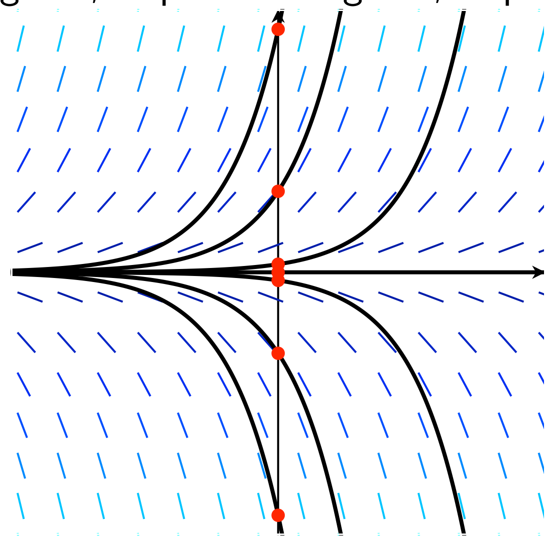
You try: The half-life of radium-226 is 1590 years. Assume radium-226 decays according to the law of natural decay, i.e. $\frac{dy}{dt} = ky$ so that $y = y(0)e^{kt}$ $(**)$, and suppose you start with 100mg of the substance.

1. What are T and $y(0)$? What are the units on y and t?
2. You have two data points — one comes from $t = 0$ that tells you $y(0)$; the other comes from $y(T) = \frac{1}{2}y(0)$. Use the equ'ns $(*)$ and $(**)$ to set up an equ'n where the only variable is k.
3. Solve for k.
4. Estimate the mass remaining after 1000 years.
5. How long will it take for the mass to be reduced to 30 mg?

What solutions look like

A differential equation like $\frac{dy}{dt} = ky$ actually says that if a solution is at height y, then its slope is $k * y$.

A slope field is a drawing of a bunch of slopes that solutions going through each point would have. For example, if $k = 1$, the corresponding slope field would have line segments of slope 1 at height 1, slope 2 at height 2, slope -1 at height -1, and so on:



(Example solutions

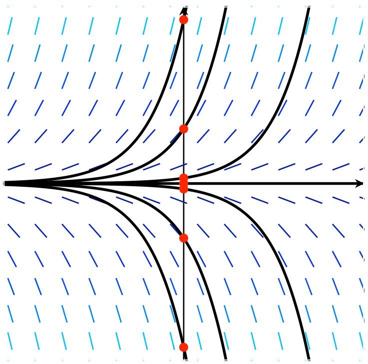
$$y = Ae^t$$

with $A = -3, -1, -\frac{1}{10}$, $0, \frac{1}{10}$, 1, and 3.)

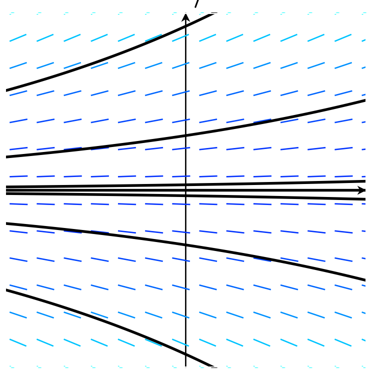
Then any solution to the equation will follow those slopes.

Example slope fields and solutions for $\frac{dy}{dx} = ky$:

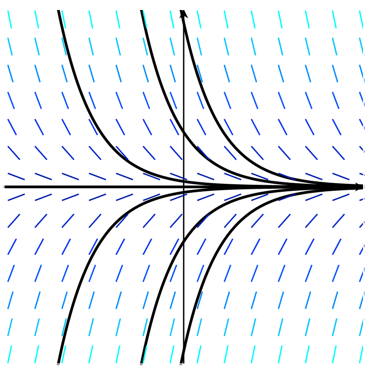
$k = 1:$



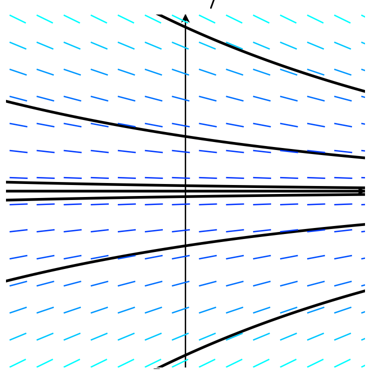
$k = 1/10:$



$k = -1:$



$k = -1/10:$

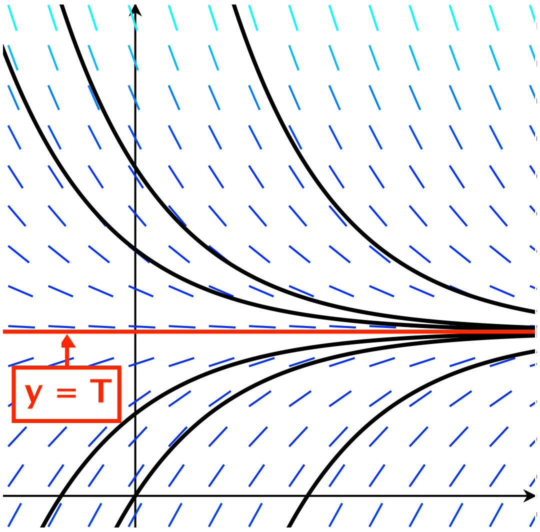


Newton's law of cooling

Newton's law of cooling states that the rate of cooling or warming of an object is proportional to the temperature difference between the object and its surroundings.

$$y = \text{temperature of object}, \qquad \frac{dy}{dt} = k(y - T),$$

where T is the temperature of the surroundings.



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To solve: Let $Y(t) = y(t) - T$, so

$$\frac{dY}{dt} = \frac{d}{dt}(y - T) = \frac{dy}{dt} = k(y - T) = kY.$$

Thus $Y(t) = Y(0)e^{kt}$. Substituting back, we have

$$y - T = (y(0) - T)e^{kt}, \text{ i.e.,} \qquad \boxed{y(t) = (y(0) - T)e^{kt} + T}.$$

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Note, if $y(0) = T$, this is the constant function $y = T$.

(For an example, see ex 3 in section 5.5)