Today: 5.4 General log and exp functions (continued)

Warm up:

$$\log_a(x) = \ln(x)/\ln(a) \qquad \frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$$

1. Evaluate the following functions.

$$\log_5(25) \qquad \log_7\sqrt{7} \qquad \log_4 8 - \log_4 2$$

2. Differentiate the following functions.

$$\log_{10} x \qquad x\log_2(x) \qquad 2^{x+\log_3(x)}$$

$$\log_5(x^2+1), \qquad x\log_e(x)-x, \qquad 3^{x\ln(x)}, \qquad \sqrt{1-(1/3)^x}$$

3. Calculate the following antiderivatives:

$$\int \frac{3^x}{3^x+3}\,dx \qquad \int \frac{2^{1/x}}{x^2}\,dx$$

$$\int e^x(3e^x+1)^{1/3}\,dx \qquad \int \frac{1}{x\ln(x)}\,dx$$

Graphs

Recall, for $a > 0$, $a^x = e^{\ln(a)x}$ and $\log_a(x) = \ln(x)/\ln(a)$.
Graphing a^x: This graph behaves like e^{rx}, with $r = \ln(a)$.

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	$0 < x < 1$	$x = 0$	$x > 1$
$\ln(x)$:	neg.	0	pos.

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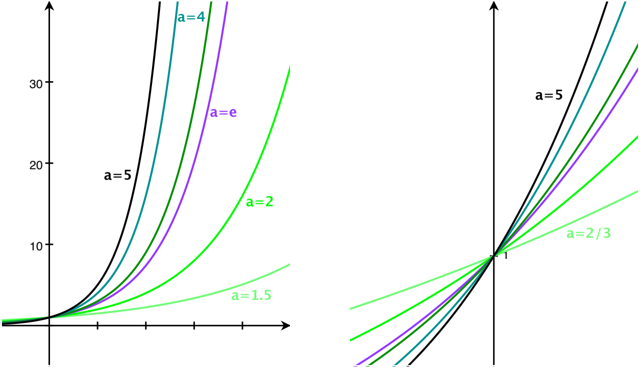
Case 1: $a > 1$. In this case, $\ln(a) > 0$. So a^x grows like e^x, e^{2x}, $e^{\frac{1}{2}x}$, etc.. Graph transformation: x-axis contraction or dilation!

Graphs

$$y = a^x, \qquad a > 1$$



Graphs

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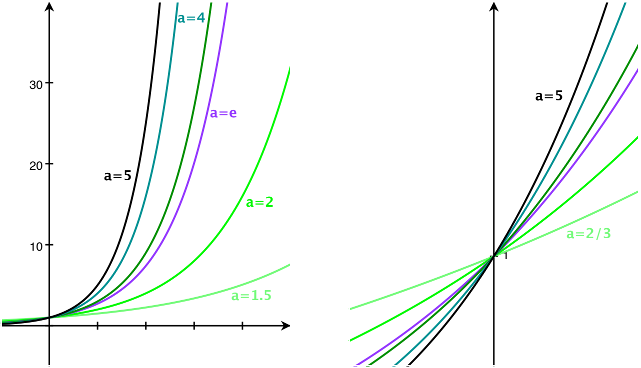

Notice: Slope at $x = 0$ is

$$\frac{d}{dx}a^x\Big|_{x=0} = \ln(a)a^x\Big|_{x=0} = \ln(a).$$

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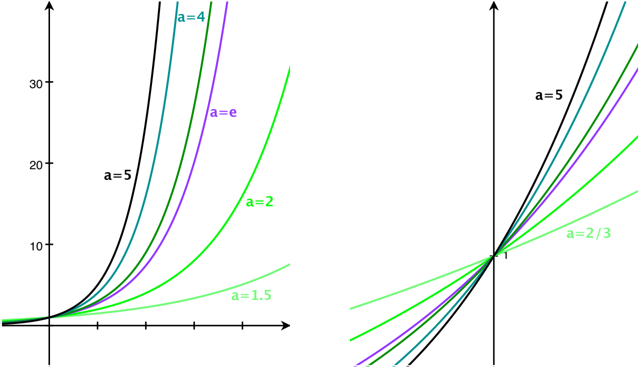

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$$\frac{d}{dx}a^x\Big|_{x=0} = \ln(a)a^x\Big|_{x=0} = \ln(a).$$

So $y = e^x$ is the exponential function whose slope through the point $(0, 1)$ is 1. For $a < e$, the slope at $x = 0$ is less than 1, and for $a > e$, the slope at $x = 0$ is greater than 1.

Graphs

Recall, for $a > 0$, $a^x = e^{\ln(a)x}$ and $\log_a(x) = \ln(x)/\ln(a)$.
Graphing a^x: This graph behaves like e^{rx}, with $r = \ln(a)$.
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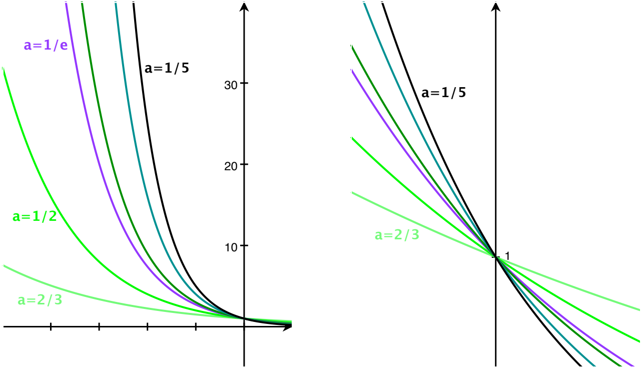
$\ln(x)$:	neg.	0	pos.

Case 1: $a > 1$. In this case, $\ln(a) > 0$. So a^x grows like e^x, e^{2x}, $e^{\frac{1}{2}x}$, etc.. Graph transformation: x-axis contraction or dilation!
Case 2: $a < 1$. In this case, $\ln(a) < 0$.

Graphs

$$y = a^x, \qquad a < 1$$



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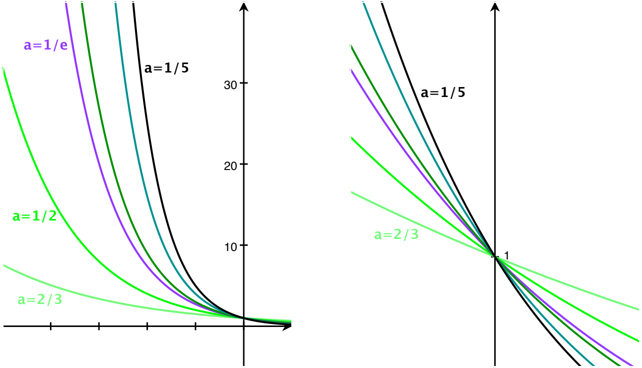

Notice:

$(1/e)^x = e^{-x}$, Graph transformation: flip over y-axis, then stretch.

Graphs

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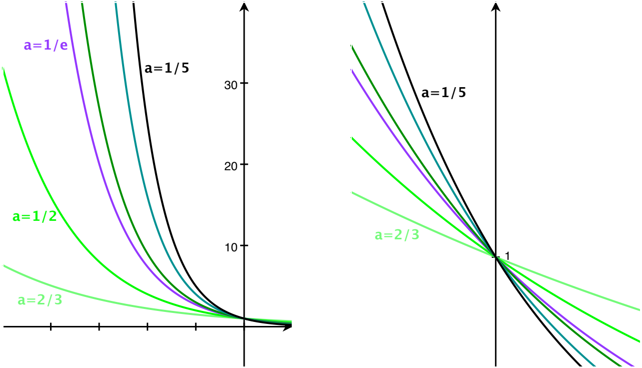

Notice:

$(1/e)^x = e^{-x}$, Graph transformation: flip over y-axis, then stretch.

So $y = (1/e)^x$ is the exponential function whose slope through the point $(0, 1)$ is -1. For $a < 1/e$, the slope at $x = 0$ is less than -1, and for $a > 1/e$, the slope at $x = 0$ is greater than -1.

Graphs

Recall, for $a > 0$, $a^x = e^{\ln(a)x}$ and $\log_a(x) = \ln(x)/\ln(a)$.
Graphing a^x: This graph behaves like e^{rx}, with $r = \ln(a)$.
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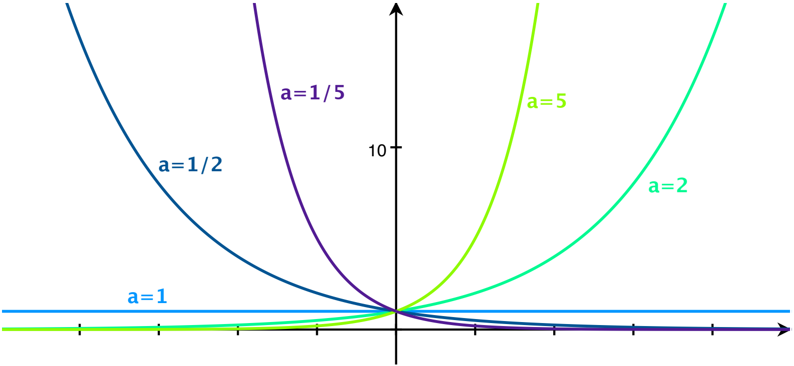
Case 1: $a > 1$. In this case, $\ln(a) > 0$. So a^x grows like e^x, e^{2x}, $e^{\frac{1}{2}x}$, etc.. Graph transformation: x-axis contraction or dilation!
Case 2: $a < 1$. In this case, $\ln(a) < 0$. So a^x grows like e^{-x}, e^{-2x}, $e^{-\frac{1}{2}x}$, etc.. Graph transformation: y-axis flip, then x-axis contraction or dilation!
Case 3: $a = 1$. This is the constant function $y = 1^x = 1$.

Graphs



$$y = a^x, \qquad \frac{dy}{dx} = \ln(a)a^x.$$

$0 < a < 1$: Slope always negative, monotonically decreasing function.

$a = 1$: Slope always zero, constant function.

$1 < a$: Slope always positive, monotonically increasing function.

You try:

Recall $\log_a(x) = \ln(x)/\ln(a)$, so that $\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}$.

Complete a similar analysis of the graphs of $y = \log_a(x)$. Namely,

- Break into cases based on a (there should be three, one of which is very strange).
- Analyze the increasing/decreasing behavior in each of the cases.
- Describe the graph transformations that take you from the graph of $\ln(x)$ to $\log_a(x)$ in each of the cases.
- Graph $y = \log_a(x)$ for a few representative values of a on the same axis (like we did on the last slide for $y = a^x$), including $a = e$ and $a = 1/e$. (Recall $2 < e < 3$.)

Check your reasoning against the fact that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other across the line $y = x$.

Continuously compounded interest and e as a limit

Say you put \$1,000 into a savings account with a 1% interest rate. How much will you have at the end of the year?

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$$\begin{aligned}
\$1,000 * (1 + .01(\tfrac{1}{12})) &= \$1,000.833 && \text{at the end of month 1,} \\
\$1,000 * (1 + .01(\tfrac{1}{12}))^2 &= \$1,001.667 && \text{at the end of month 2,} \\
\$1,000 * (1 + .01(\tfrac{1}{12}))^3 &= \$1,002.502 && \text{at the end of month 3,} \\
&\vdots \\
\$1,000 * (1 + .01(\tfrac{1}{12}))^{12} &= \$1,010.046 && \text{at the end of month 12.,}
\end{aligned}$$

i.e. a whole 4.6 cents more than if it had been compounded at the end of the year.

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Similarly, if that 1% is compounded weekly, you'll have

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or compounded daily, you'll have

$$\$1,000 * (1 + .01(\tfrac{1}{365}))^{365} = \$1,010.050 \text{ at the end of the year,}$$

and so on.

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Say you put \$1,000 into a savings account with a 1% interest rate.

How much will you have at the end of the year?

If the interest is compounded n times a year, then you'll have

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In general, if the deposited amount is D, the interest rate is r, and the number of times it's compounded is n, you end up with

$$D(1 + r(\tfrac{1}{n}))^n \text{ at the end of the year.}$$

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Back to logarithms: Let $f(x) = \ln(x)$.

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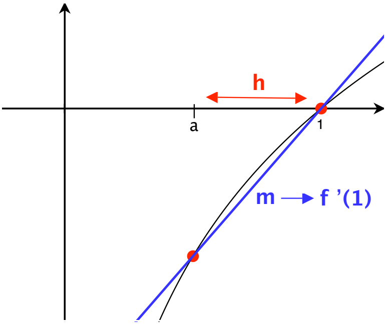

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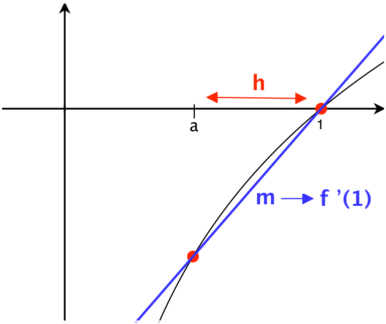

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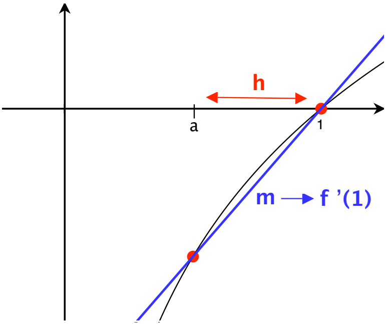

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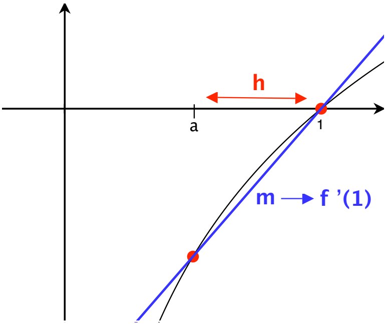

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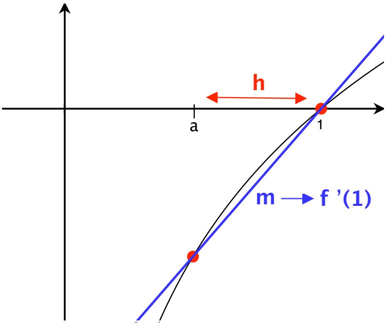

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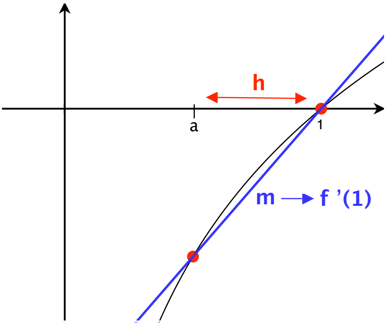

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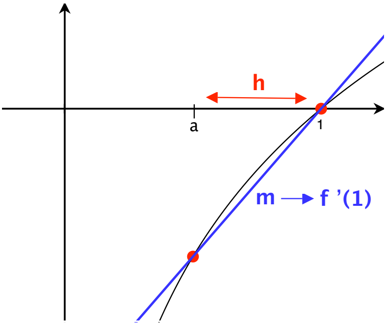

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&= \lim_{h\to 0}\ln\left((1+h)^{1/h}\right)\\
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\end{aligned}
$$

Since $1=\ln\left(\lim_{h\to 0}(1+h)^{1/h}\right)$, we have $\boxed{e=\lim_{h\to 0}(1+h)^{1/h}}$.

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$$y(0) = Ce^0 = C.$$

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Let y be the amount of money we have in the bank at time t. We saw before that if we deposit $D = y(0)$ into an account that accrues interest at a rate r, compounded continuously, then we'll have

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So (1) $y(1993 - 1950) = y(43) \approx 2560e^{(.0172)43} \approx 5360$ million, and (2) $y(2020 - 1950) = y(70) \approx 2560e^{(.0172)70} \approx 8520$ million.

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You try: The half-life of radium-226 is 1590 years. Assume radium-226 decays according to the law of natural decay, i.e. $\frac{dy}{dt} = ky$ so that $y = y(0)e^{kt}$ $(**)$, and suppose you start with 100mg of the substance.

1. What are T and $y(0)$? What are the units on y and t?
2. You have two data points — one comes from $t = 0$ that tells you $y(0)$; the other comes from $y(T) = \frac{1}{2}y(0)$. Use the equ'ns $(*)$ and $(**)$ to set up an equ'n where the only variable is k.
3. Solve for k.
4. Estimate the mass remaining after 1000 years.
5. How long will it take for the mass to be reduced to 30 mg?

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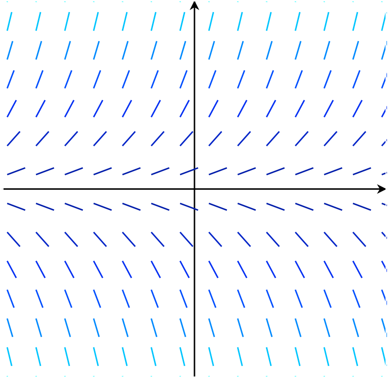

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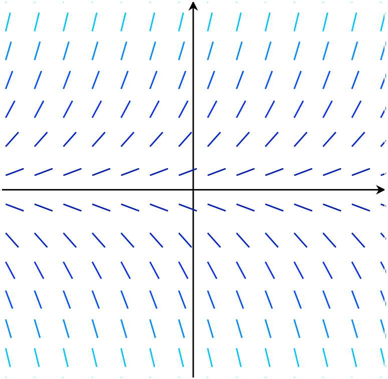

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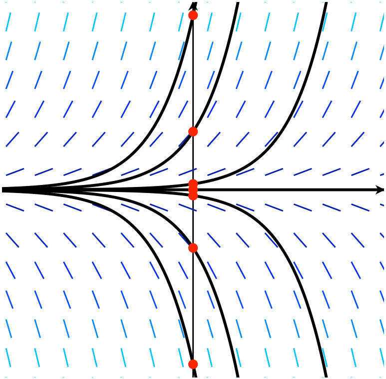

(Example solutions

$$y = Ae^t$$

with $A = -3, -1, -\frac{1}{10}$, $0, \frac{1}{10}, 1$, and $3.$)

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Example slope fields and solutions for $\frac{dy}{dx} = ky$:

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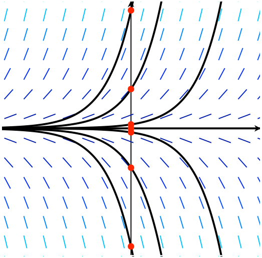

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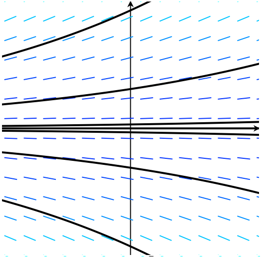

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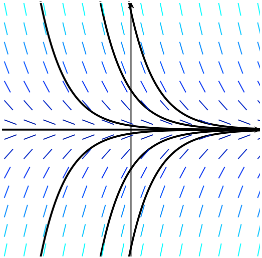

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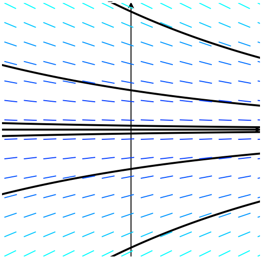

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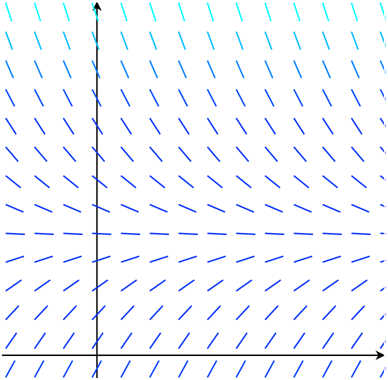

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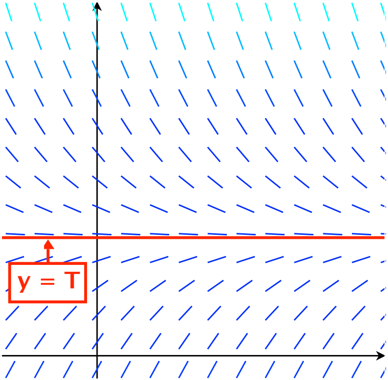

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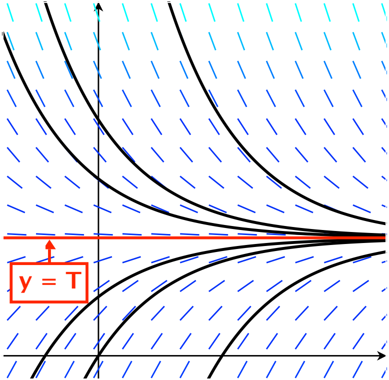

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Newton's law of cooling states that the rate of cooling or warming of an object is proportional to the temperature difference between the object and its surroundings.

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Note, if $y(0) = T$, this is the constant function $y = T$.