Today: 5.3 The Natural Exponential Function (continued)

Warm up:

Recall that e is the number defined by $\ln(e) = 1$, so that

$$\ln(y) = x \qquad \text{if any only if} \qquad y = e^x.$$

Further, recall $\ln(ab) = \ln(a) + \ln(b)$ and $\ln(a^p) = p\ln(a)$.

Solve the following equations for x.

(1) $\ln(x) = 10$

(2) $e^x = 3$

(3) $10e^{17-6x} + 5 = 25$

(4) $e^{2\ln(x)+\ln(5x)} = 40$

Exponential functions facts

Recall from last time:	Corresp. exp facts:	Why:
$\ln(ab) = \ln(a) + \ln(b)$	$e^{A+B} = e^A e^B$	$\ln(e^A e^B) = \cdots$
$\ln(a^p) = p\ln(a)$	$(e^A)^P = e^{PA}$	$\ln((e^A)^P) = \cdots$
$\ln(a/b) = \ln(a) - \ln(b)$	$e^{A-B} = e^A/e^B$	$e^{A-B} = e^A e^{-B} = \cdots$
$\lim_{x\to 0^+} \ln(x) = -\infty$	$\lim_{x\to-\infty} e^x = 0$	$e^x = y \Leftrightarrow x = \ln(y)$
$\lim_{x\to\infty} \ln(x) = \infty$	$\lim_{x\to\infty} e^x = \infty$	$e^x = y \Leftrightarrow x = \ln(y)$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\frac{d}{dx}e^x = e^x$	Logarithmic diff'n

Let $y = e^x$. Then $\ln(y) = x$. Thus

$$\frac{d}{dx}(\text{LHS}) = \ln(y) = \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\text{RHS})\frac{d}{dx}x = 1.$$

So

$$\frac{dy}{dx} = y = e^x.$$

You try:

$$e^{A+B} = e^A e^B, \quad (e^A)^P = e^{PA}, \quad \frac{d}{dx} e^x = e^x, \quad \int e^x \, dx = e^x + C$$

1. Differentiate the following functions.

$$e^{\cos(x)}, \qquad e^{3x^2}\sin(x), \qquad e^{3x}\ln(\sin(x)).$$

2. Find the antiderivative of the following functions.

$$e^{3x}, \qquad xe^{x^2}, \qquad \frac{e^{\sqrt{1+x}}}{\sqrt{1+x}}.$$

3. Sketch a graph of xe^x.
   (Calculate intervals of pos./neg., incr./decr., concave up/down.)

WARNING: Study problems 5.3 #11–14. On syllabus, but not on webassign! Use graph transformations, not calculus.

Limits

$$\boxed{\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty}$$

Example:

$$\lim_{x\to\infty} e^{-x} = \lim_{y\to-\infty} e^y = 0 \qquad \text{(let } y = -x\text{)}$$

$$\lim_{x\to\infty} \frac{e^x - 1}{e^x + 1} = \lim_{x\to\infty} \frac{e^x - 1}{e^x + 1}\left(\frac{e^{-x}}{e^{-x}}\right) \lim_{x\to\infty} \frac{1 - e^{-x}}{1 + e^{-x}} = \frac{1-0}{1+0} = 1.$$

You try:

Calculate the following limits:

$$\lim_{x\to\infty} \frac{6e^x + e^{-x}}{2e^x + 1}$$

$$\lim_{x\to 5^-} e^{1/(x-5)} \qquad\qquad \lim_{x\to-\infty} e^{x^2}$$

WARNING: Study problems 5.3 #17–20. On syllabus, but not on webassign!

5.4: General logarithmic and exponential functions

Fix a number $a > 0$. We can already say things like

$$a^2 = a * a, \qquad a^3 = a * a * a, \qquad \text{and so on.}$$

We've defined things like

$$a^{-1} = 1/a, \qquad a^{1/2} = \sqrt{a}, \qquad \text{and so on,}$$

and found rules like

$$a^p a^q = a^{p+q}, \qquad (a^p)^q = a^{pq}, \qquad \text{and so on.}$$

So I know things like

$$a^{4/3} = (a^{1/3})^4 = \sqrt[3]{a} * \sqrt[3]{a} * \sqrt[3]{a} * \sqrt[3]{a}.$$

But what could I possibly mean by something like a^π???

Answer: Since

$$e^{AB} = (e^A)^B \qquad \text{and} \qquad e^{\ln(A)} = A,$$

we can *define*

$$a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)x}.$$

Exponential function with base a

For $a > 0$, define

$$a^x = e^{\ln(a)x}.$$

This definition still preserves all the rules we know and love, like

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Why? For example,

$$a^{x+y} = e^{\ln(a)(x+y)} = e^{\ln(a)x + \ln(a)y} = e^{\ln(a)x} e^{\ln(a)y} = a^x a^y.$$

Derivatives: Using chain rule, we have

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{\ln(a)x} = \ln(a) e^{\ln(a)x} = \ln(a) a^x.$$

Integrals: Similarly, using substitution, we have

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C.$$

You try:

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x,$$
$$\frac{d}{dx} a^x = \ln(a) a^x, \quad \text{and} \quad \int a^x \, dx = \frac{a^x}{\ln(a)} + C.$$

1. Write the following expressions as e^{stuff} (simplify).

$$2^x, \qquad x^5, \qquad \pi^2$$

2. Differentiate the following functions.

$$2^x, \qquad 3^{\sin(x)}, \qquad (1/2)^{\sin(x) - 5\ln(x)}$$

3. Find the antiderivative of the following functions.

$$3^x, \qquad 2^x \cos(2^x), \quad \frac{5^x}{5^x - 1}$$

WARNING: 5.4 #3, 5, 21–27 (odd), 39 on syllabus but not on webassign.

General logarithmic function

Since e^x is invertible, so is e^{rx}, and thus so is a^x. Define $\log_a(x)$ by

$$a^y = x \qquad \text{if and only if} \qquad y = \log_a(x).$$

So, for example, since

$$\log_{10}(10^3) = 3, \quad 3^{\log_3(5)} = 5, \qquad \text{etc..}$$

Note that $x = a^y$ also implies (take natural log of both sides)

$$\ln(x) = \ln(a^y) = \ln(e^{\ln(a)y}) = \ln(a)y,$$

so

$$\boxed{\log_a(x) = y = \ln(x)/\ln(a).}$$

Thus $\log_a(x)$ satisfies the same algebraic rules as $\ln(x)$, and

$$\boxed{\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}}.$$

You try:

$$\boxed{\log_a(x) = \ln(x)/\ln(a) \qquad \frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)x}}$$

1. Evaluate the following functions.

$$\log_5(25) \qquad \log_7\sqrt{7} \qquad \log_4 8 - \log_4 2$$

2. Differentiate the following functions.

$$\log_{10} x \qquad x\log_2(x) \qquad 2^{x+\log_3(x)}$$

3. Sketch graphs of the following 3 functions on the same set of axes.

$$y = \ln(x), \qquad y = \log_2(x), \qquad y = \log_3(x).$$

(Recall $2 < e < 3$.)