5.2 The Natural Logarithmic Function

Reminders:

1. Sign up through WebAssign for homework.
   Course key: ccny 4222 6935
   First two assignments due next Wednesday.
2. Email me at zdaugherty@gmail.com from your preferred email address, subject line "Math 202 FG" with your full name and why you are in this class (be specific). If you want to help me learn your name, please include a recognizable picture of you.

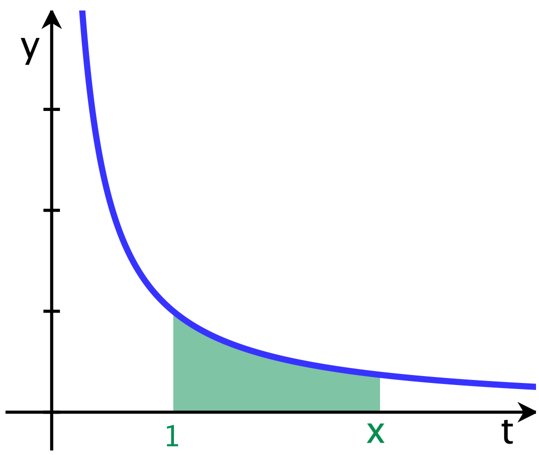
Warmup: Recall $\frac{d}{dx}\sin(x) = \cos(x)$. Differentiate the following functions.

$$\sin(x^3), \quad x^3\sin(x), \quad x^3\sin(x^{-1} + 3x^5)$$

Definition: $\ln(x)$

Define the natural logarithmic function by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$



Recall some facts about definite integrals:

For a function $f(t)$ that is integrable on $[a, b]$ $(a \leq b)$, we have the following.

1. The definite integral

$$\int_a^b f(t)\ dt \quad \text{evaluates to the signed area}$$

   between $f(t)$ and the t axis, between a and b.

   Signed area means that it takes a negative value if it falls below the t-axis.

2. For any c in $[a, b]$,

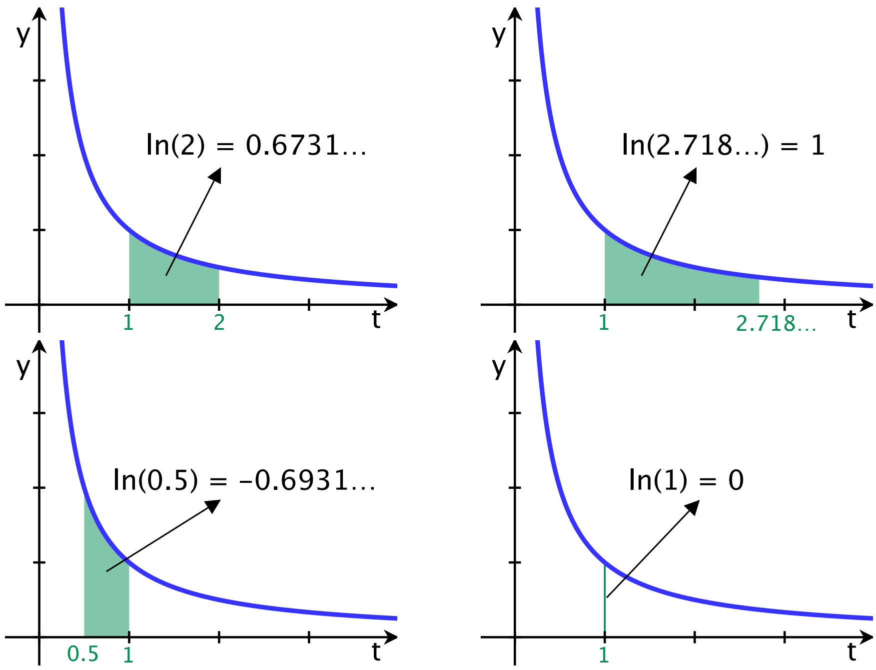
$$\int_c^c f(t)\ dt = 0.$$

3. Reversing the order of integration gives

$$\int_b^a f(t)\ dt = -\int_a^b f(t)\ dt.$$

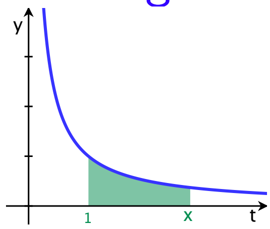
Some examples of $\ln(x)$

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$



Back to some facts about definite integrals:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



1. The definite integral

$$\int_a^b f(t)\,dt \quad \text{evaluates to the signed area...}$$

   Conclusion: $\boxed{\text{For } x > 1,\ \ln(x) > 0.}$

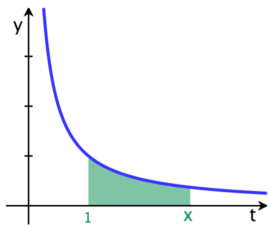
2. For any c in $[a, b]$, $\int_c^c f(t)\,dt = 0$. Conclusion: $\boxed{\ln(1) = 0}$
3. Reversing the order of integration gives

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

   Conclusion: $\boxed{\text{For } 0 < x < 1,\ \ln(x) < 0.}$

Some facts about $\ln(x)$

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



1. $\ln(x) < 0$ for $0 < x < 1$, $\ln(x) = 0$ at $x = 1$, $\ln(x) > 0$ for $x > 1$, and $\ln(x)$ is undefined for $x \leq 0$.
2. By the fundamental theorem of calculus,

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

   So, for example, $\ln(x)$ is monotonically increasing since $1/x > 0$ for $x > 0$.
3. We have the following algebraic properties:

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{and} \quad \ln(a^p) = p\ln(a)$$

   These both follow from the fact that $\frac{d}{dx}\ln(x) = 1/x$.

You try:

1. Use the algebraic rules

$$\ln(ab) = \ln(a) + \ln(b) \quad \ln(a^p) = p\ln(a) \quad \ln(a/b) = \ln(a) - \ln(b)$$

to expand the expressions
(write in terms of a bunch of $\ln(f(x))$'s where $f(x)$ is as simple as possible)

$$\ln\left((x^2+2)^3\right), \quad \ln\left(\frac{(x^2+2)^3}{5x+5}\right), \text{ and } \ln\left(\frac{(x^2+2)^3\sin(x)}{5x+5}\right),$$

and to contract the expressions (write in terms of one $\ln(\dots)$)

$$\ln(x) + \ln(2), \quad 3\ln(x) - \ln(2), \text{ and } 5\left(3\ln(x) - \ln(2)\right).$$

2. Differentiate the following functions. Simplify where you can.

$$\ln(x^3+2), \quad \ln(\sin(x)), \quad \ln\left(\frac{x+1}{\sqrt{x-2}}\right).$$

(Hint: Lots of chain rule!! $\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}f'(x)$)

Derivatives with absolute values

Example: Calculate $\frac{d}{dx}\ln|x|$.
Recall

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

So (1) the domain of $\ln|x|$ is $(-\infty, 0) \cup (0, \infty)$, and (2)

$$\ln|x| = \begin{cases} \ln(x) & x \geq 0, \\ \ln(-x) & x < 0. \end{cases}$$

So

$$\frac{d}{dx}\ln|x| = \begin{cases} 1/x & x \geq 0, \\ -(1/(-x)) = 1/x & x < 0, \end{cases} = 1/x.$$

Therefore,

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

(Nice to know, since $1/x$ is defined over all real numbers $\neq 0$, but $\ln(x)$ is only defined over positive real numbers!)

Reviewing u-substitution

Example: Calculate $\int \tan(x)\ dx$.

Recall $\tan(x) = \sin(x)/\cos(x)$. So

$$\int \tan(x)\ dx = \int \frac{\sin(x)}{\cos(x)} dx \qquad \text{Let } u = \cos(x)$$

So $du = -\sin(x)dx$.

$$\begin{aligned} &= -\int \frac{1}{u}\ du \\ &= -\ln|u| + C \\ &= -\ln|\cos(x)| + C \\ &= \ln|\sec(x)| + C \qquad \text{since } |\cos(x)|^{-1} = |\sec(x)|. \end{aligned}$$

You try: calculate

$$\int \frac{x}{x^2+1}\ dx, \quad \int \cot(x)\ dx, \quad \int \frac{\ln(x)}{x}\ dx.$$

Logarithmic differentiation

Example: Calculate the derivative of $y = \frac{(x^2+2)^3 \sin(x)}{5x+5}$

Step 1: Take logarithms of both sides.

$$\ln(y) = \ln\left(\frac{(x^2+2)^3 \sin(x)}{5x+5}\right)$$

Step 2: Use algebraic log rules to expand.

Before, we showed that

$$\ln\left(\frac{(x^2+2)^3 \sin(x)}{5x+5}\right) = 3\ln(x^2+2)+\ln(\sin(x))-\ln(x+1)-\ln(5).$$

Step 3: Take $\frac{d}{dx}$ of both sides using implicit differentiation.

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{2x}{x^2+2} + \frac{\cos(x)}{\sin(x)} - \frac{1}{x+1} - 0.$$

Step 4: Solve for $\frac{dy}{dx}$ and substitute for y.

$$\frac{dy}{dx} = \left(3\frac{2x}{x^2+2} + \frac{\cos(x)}{\sin(x)} - \frac{1}{x+1}\right) \frac{(x^2+2)^3 \sin(x)}{5x+5}.$$

Logarithmic differentiation

Sometimes, we need logarithmic differentiation to calculate derivatives at all!

Example: Calculate the derivative of $y = x^x$.

Step 1: Take logarithms of both sides.

$$\ln(y) = \ln(x^x) = x\ln(x)$$

Step 2: Use algebraic log rules to expand.

Step 3: Take $\frac{d}{dx}$ of both sides using implicit differentiation.

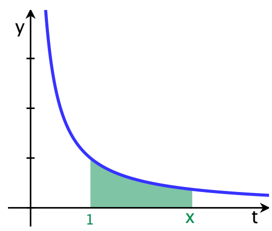
$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + x\frac{1}{x} = \ln(x) + 1$$

Step 4: Solve for $\frac{dy}{dx}$ and substitute for y.

$$\frac{dy}{dx} = (\ln(x) + 1)y = (\ln(x) + 1)x^x.$$

Some facts about $\ln(x)$

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



1.

	$0 < x < 1$	$x = 0$	$x > 1$
$\ln(x)$:	neg.	0	pos.

2. $\frac{d}{dx}\ln(x) = 1/x$.
3. $\ln(ab) = \ln(a) + \ln(b)$ and $\ln(a^p) = p\ln(a)$.
4. We have the limits

$$\lim_{x\to\infty} \ln(x) = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} = -\infty.$$

These follow from setting $x = 2^r$, and letting $r \to \pm\infty$.

Graphing $\ln(x)$

Recall graph sketching techniques:

1. Calculate domain, plot points. Especially, plot roots, and evaluate pos/neg intervals.

   Domain: $(0, \infty)$,

	$0 < x < 1$	$x = 0$	$x > 1$
$\ln(x)$:	neg.	0	pos.

2. Calculate limits as x goes to boundaries of the domain.

   $$\lim_{x\to\infty} \ln(x) = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} = -\infty$$

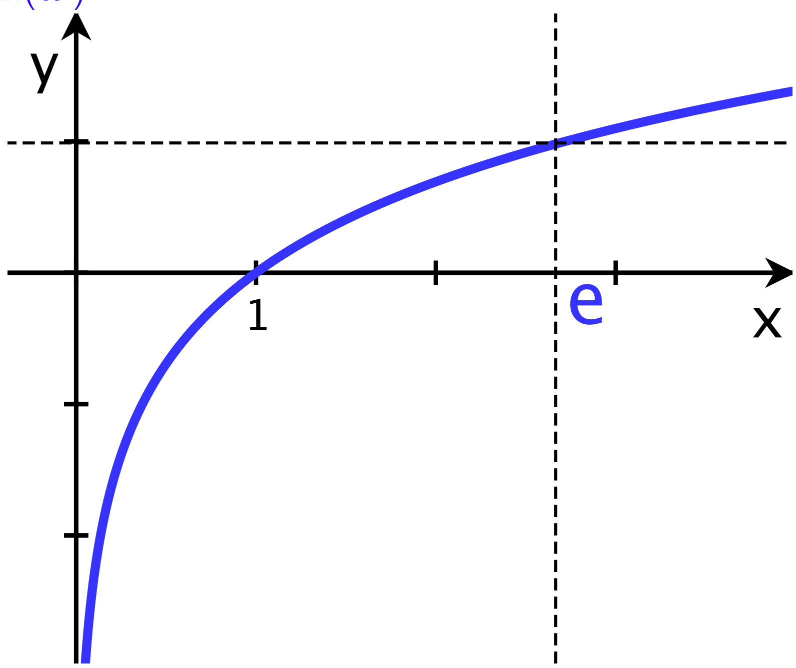
3. Calculate first derivative and evaluate pos/neg intervals. (Increasing/decreasing)

   $\frac{d}{dx}\ln(x) = 1/x > 0$ for $x > 0$: always increasing

4. Calculate second derivative and evaluate pos/neg intervals. (Concave up/down)

   $\frac{d^2}{dx^2}\ln(x) = -1/x^2 < 0$ for $x > 0$: always concave down

Graphing $\ln(x)$



Define the number e by $\ln(e) = 1$
(such a number exists by the intermediate value theorem).

$$\boxed{e = 2.718\ldots}$$

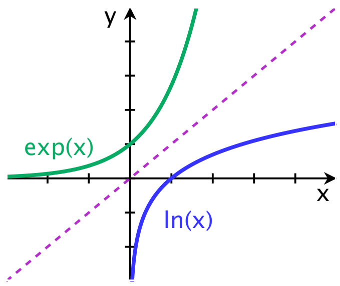
Section 5.3: The natural exponential function

Note that since $\ln(x)$ is always increasing, it is one-to-one, and therefore invertible! Define $\exp(x)$ as the inverse function of $\ln(x)$, e.g.

$$\exp(x) = y \quad \text{if and only if} \quad x = \ln(y). \qquad (*)$$

Some facts about $\exp(x)$:

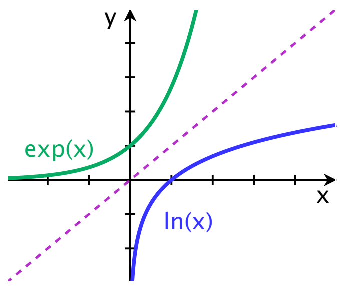
1. Since $\ln(1) = 0$, we have $\exp(0) = 1$. Similarly, $\ln(e) = 1$ implies $\exp(1) = e$.
2. Domain: (range of $\ln(x)$) $(-\infty, \infty)$
   Range: (domain of $\ln(x)$) $(0, \infty)$
3. Graph:



Section 5.3: The natural exponential function

Some facts about $\exp(x)$:

1. $\exp(0) = 1, \exp(1) = e$.
2. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$.
3. Graph:



4. We have $\ln(e^x) = x\ln(e) = x * 1 = x$, and so $(*)$ gives

$$\ln(e^x) = x \quad \text{implies} \quad \boxed{\exp(x) = e^x}$$

So

$$e^{\ln(x)} = x \text{ for } x > 0, \quad \text{and} \quad \ln(e^x) = x \text{ for all } x.$$

Exercise: Use logarithmic differentiation to calculate $\frac{d}{dx}e^x$.