5.2 The Natural Logarithmic Function

Reminders:

1. Sign up through WebAssign for homework.
   Course key: ccny 4222 6935
   First two assignments due next Wednesday.
2. Email me at zdaugherty@gmail.com from your preferred email address, subject line "Math 202 FG" with your full name and why you are in this class (be specific). If you want to help me learn your name, please include a recognizable picture of you.

Warmup: Recall $\frac{d}{dx}\sin(x) = \cos(x)$. Differentiate the following functions.

$$\sin(x^3), \quad x^3\sin(x), \quad x^3\sin(x^{-1}+3x^5)$$

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Answers:

$$3x^2\cos(x^3), \quad 3x^2\sin(x) + x^3\sin(x), \text{ and}$$

$$3x^2\sin(x^{-1}+3x^5) + x^3(-x^{-2}+15x^4)\cos(x^{-1}+3x^5),$$

respectively.

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Define the natural logarithmic function by

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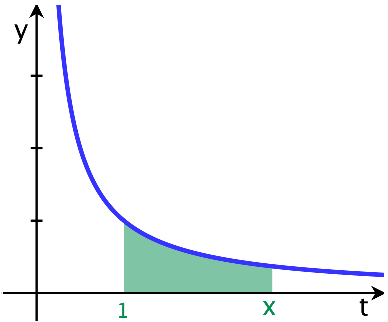

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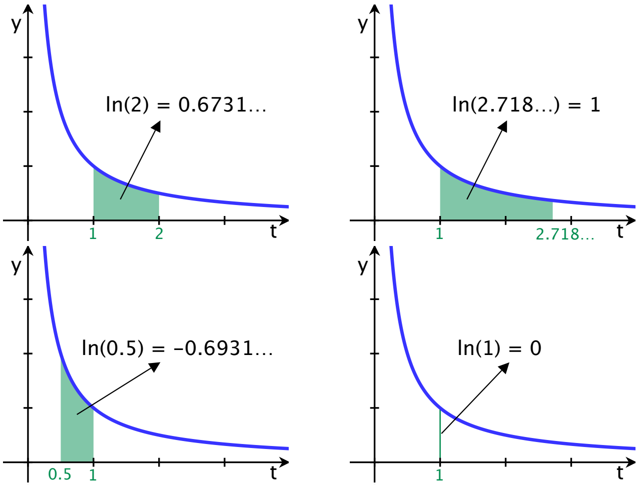
Some examples of $\ln(x)$

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$



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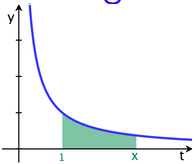

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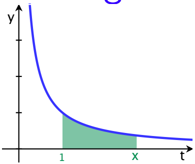

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Conclusion: For $x > 1$, $\ln(x) > 0$.

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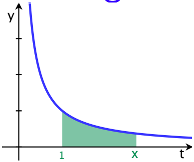

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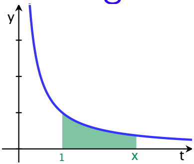

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Conclusion: $\boxed{\text{For } 0 < x < 1,\ \ln(x) < 0.}$

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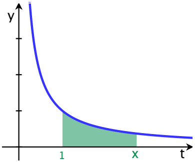

1. $\ln(x) < 0$ for $0 < x < 1$, $\ln(x) = 0$ at $x = 1$, $\ln(x) > 0$ for $x > 1$, and $\ln(x)$ is undefined for $x \leq 0$.

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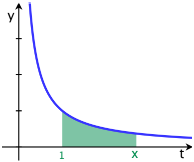

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2. By the fundamental theorem of calculus,

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\ dt = \frac{1}{x}.$$

So, for example, $\ln(x)$ is monotonically increasing since $1/x > 0$ for $x > 0$.

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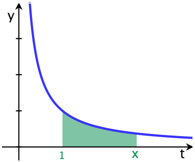

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3. We have the following algebraic properties:

$$\ln(ab) = \ln(a) + \ln(b) \quad \text{and} \quad \ln(a^p) = p\ln(a)$$

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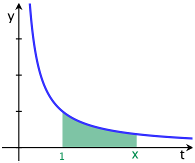

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These all follow from the fact that $\frac{d}{dx}\ln(x) = 1/x$. So, for example,

$$\ln(a/b) = \ln(ab^{-1}) = \ln(a) + \ln(b^{-1}) = \ln(a) - \ln(b).$$

You try:

1. Use the algebraic rules

   $$\ln(ab) = \ln(a) + \ln(b) \quad \ln(a^p) = p\ln(a) \quad \ln(a/b) = \ln(a) - \ln(b)$$

   to expand the expressions
   (write in terms of a bunch of $\ln(f(x))$'s where $f(x)$ is as simple as possible)

   $$\ln\left((x^2+2)^3\right), \quad \ln\left(\frac{(x^2+2)^3}{5x+5}\right), \text{ and } \ln\left(\frac{(x^2+2)^3\sin(x)}{5x+5}\right),$$

   and to contract the expressions (write in terms of one $\ln(\dots)$)

   $$\ln(x) + \ln(2), \quad 3\ln(x) - \ln(2), \text{ and } 5\left(3\ln(x) - \ln(2)\right).$$

2. Differentiate the following functions. Simplify where you can.

   $$\ln(x^3+2), \quad \ln(\sin(x)), \quad \ln\left(\frac{x+1}{\sqrt{x-2}}\right).$$

   (Hint: Lots of chain rule!! $\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}f'(x)$)

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Therefore,

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

(Nice to know, since $1/x$ is defined over all real numbers $\neq 0$, but $\ln(x)$ is only defined over positive real numbers!)

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You try: calculate

$$\int \frac{x}{x^2+1}\ dx, \quad \int \cot(x)\ dx, \quad \int \frac{\ln(x)}{x}\ dx.$$

Logarithmic differentiation

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$$\ln(y) = \ln(x^x) = x\ln(x)$$

Step 2: Use algebraic log rules to expand.

Step 3: Take $\frac{d}{dx}$ of both sides using implicit differentiation.

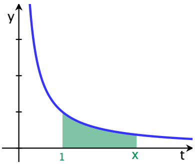
$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + x\frac{1}{x} = \ln(x) + 1$$

Step 4: Solve for $\frac{dy}{dx}$ and substitute for y.

$$\frac{dy}{dx} = (\ln(x) + 1)y = (\ln(x) + 1)x^x.$$

Some facts about $\ln(x)$

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$



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	$0 < x < 1$	$x = 0$	$x > 1$
$\ln(x)$:	neg.	0	pos.

2. $\frac{d}{dx}\ln(x) = 1/x$.
3. $\ln(ab) = \ln(a) + \ln(b)$ and $\ln(a^p) = p\ln(a)$.

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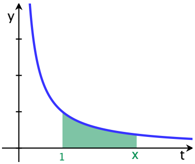

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4. We have the limits

$$\lim_{x\to\infty} \ln(x) = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} = -\infty.$$

These follow from setting $x = 2^r$, and letting $r \to \pm\infty$.

Graphing $\ln(x)$

Recall graph sketching techniques:

1. Calculate domain, plot points. Especially, plot roots, and evaluate pos/neg intervals.
2. Calculate limits as x goes to boundaries of the domain.
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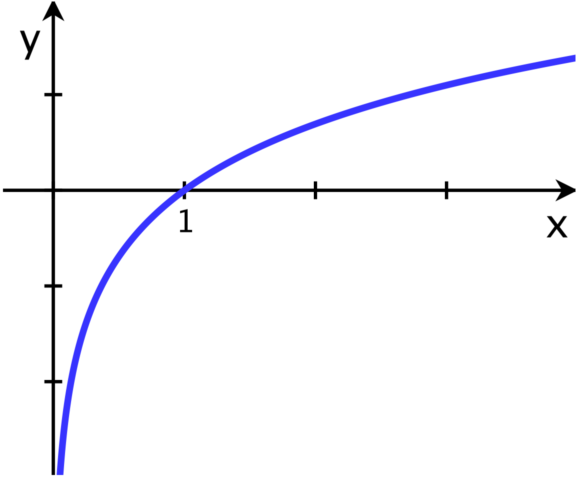

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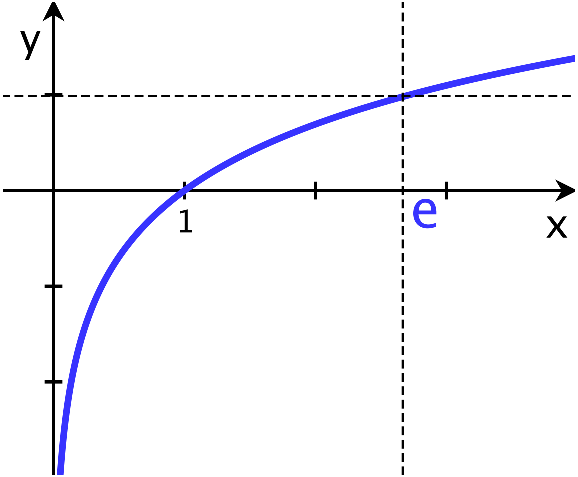

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(such a number exists by the intermediate value theorem).

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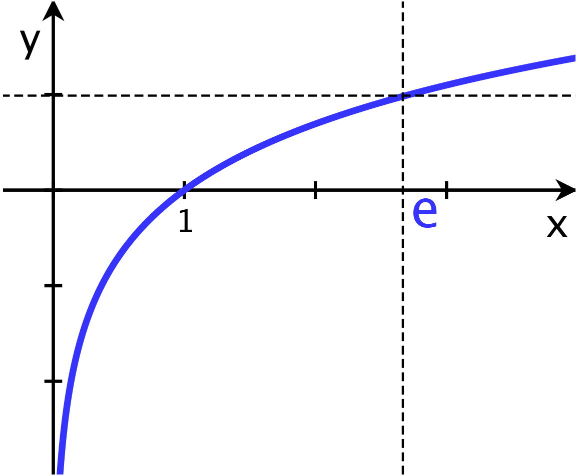

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$$\boxed{e = 2.718\ldots}$$

Section 5.3: The natural exponential function

Note that since $\ln(x)$ is always increasing, it is one-to-one, and therefore invertible!

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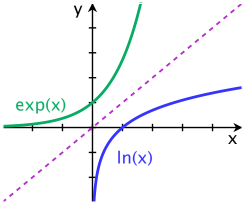

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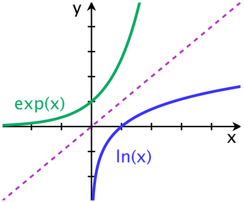

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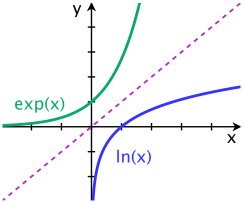

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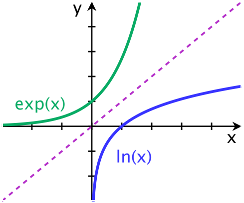

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Exercise: Use logarithmic differentiation to calculate $\frac{d}{dx} e^x$.