Logistics:

Professor Zajj Daugherty, NAC 6-301
Office hours: Monday 2:30–3:30, Wednesday 11–12.
For now, the section's website is at

http://math.sci.ccny.cuny.edu/pages?name=m202FGs16

Grades: You grades will be based on

1. 15%: Homework and occasional quizzes:
   Homework though WebAssign (www.webassign.net)
   Course key: ccny 4222 6935
   Online homework due 1 week after content is covered.
2. 45%: Midterms
   Midterms will be in class, and are tentatively scheduled for Wednesday 3/16 and Wednesday 4/20.
3. 40%: Course-wide final
   The final will be modeled after the homework list on the course-wide syllabus. Most, but not all, will appear on your WebAssign homeworks.

This course will cover. . .

- Chapter 5: Inverse functions (exponential, logarithmic, inverse trig, and hyperbolic functions; indeterminate forms and L'Hospital's rule)
- Chapter 6: Techniques of integration (calculating integrals algebraically, educated guessing); skip: 6.4.
- Chapter 7: Applications of integration (area, volume, arc length, physics); skip: 7.5, 7.7.
- Chapter 9: Parametric equations and polar coordinates; skip 9.5.
- Other stuff: Conic sections

Recall that a function is a machine that takes a number from one set and puts a number of another set. Must be well-defined, meaning the function is decisive: (1) always has an answer and (2) always puts out one answer for each number taken in.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$; e.g.

$$\begin{array}{ccccc} 1 & -2 & -\pi & 1/3 & \text{etc.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 4 & \pi^2 & 1/9 & \end{array}$$

2. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto |\sqrt{x}|$; e.g.

$$\begin{array}{ccccc} 1 & 4 & \pi^2 & 1/9 & \text{etc.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2 & \pi & 1/3 & \end{array}$$

Note that $\sqrt{x}$ is only a function when we go to extra effort to decide that we're always going to choose the positive answer.

3. Let bacteria grow, and measure population over time. Consider $N : \mathbb{N} \to \mathbb{N}$ by $N(t) = \#$ bacteria at time t.

t (hours)	$N(t) =$ pop. at time t
0	100
1	168
2	259
3	258
4	445
5	509

Now suppose we we're trying to ask the question "how long will it take to grow at least 500 bacteria?"

Answer: between 4 and 5 hours

Inverse functions

Given a function f, the *inverse function* f^{-1} is the machine that takes in f's output, and returns the corresponding input.

$$x \xmapsto{f} f(x) \xmapsto{f^{-1}} x$$

In notation, we write that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

Example: If $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

$$f(x) = x^2, \qquad \text{then} \qquad f^{-1}(x) = |\sqrt{x}| = \sqrt{x}.$$

Non-Example: If $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by

$$f(x) = x^2, \qquad \text{then} \qquad f^{-1}(x) \text{ is not well-defined.}$$

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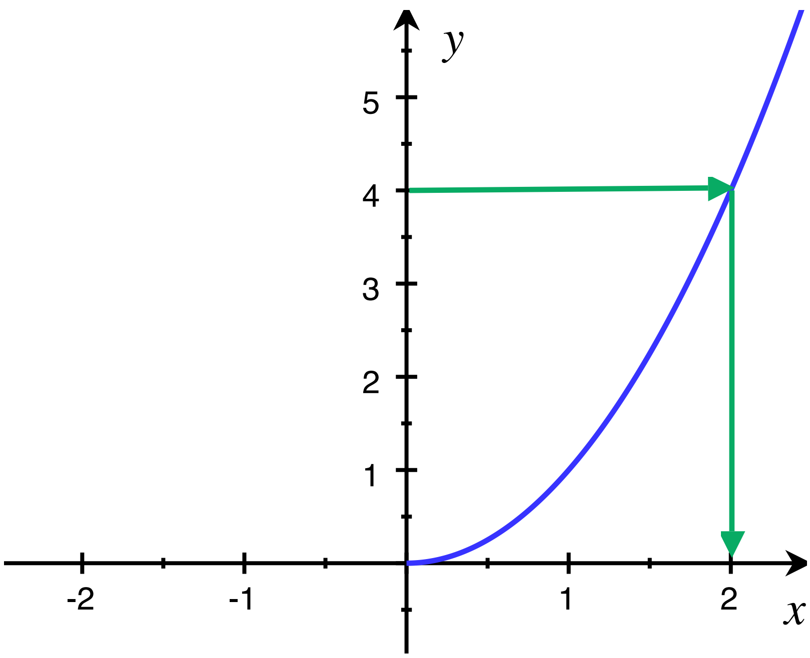

If $y = x^2$ and $x \geq 0$, then $x = |\sqrt{y}|$.

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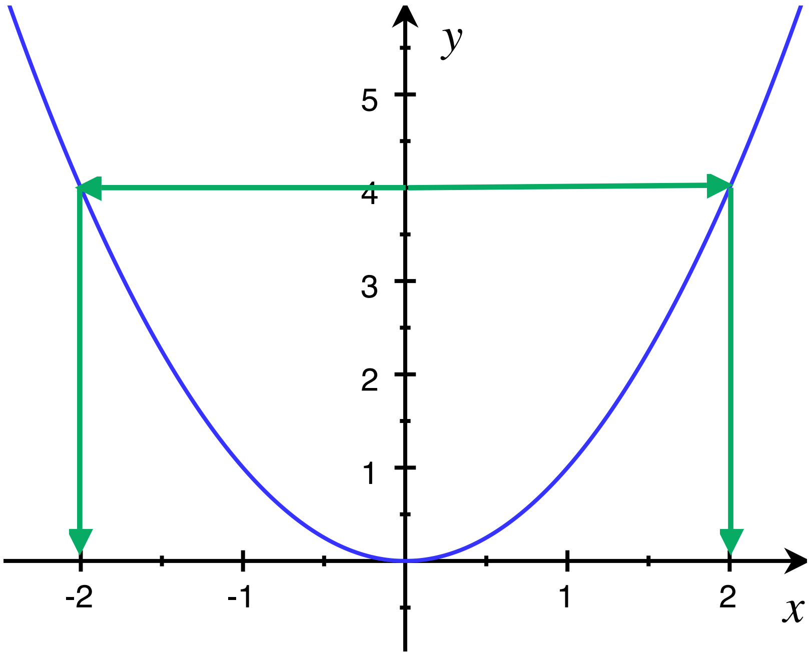
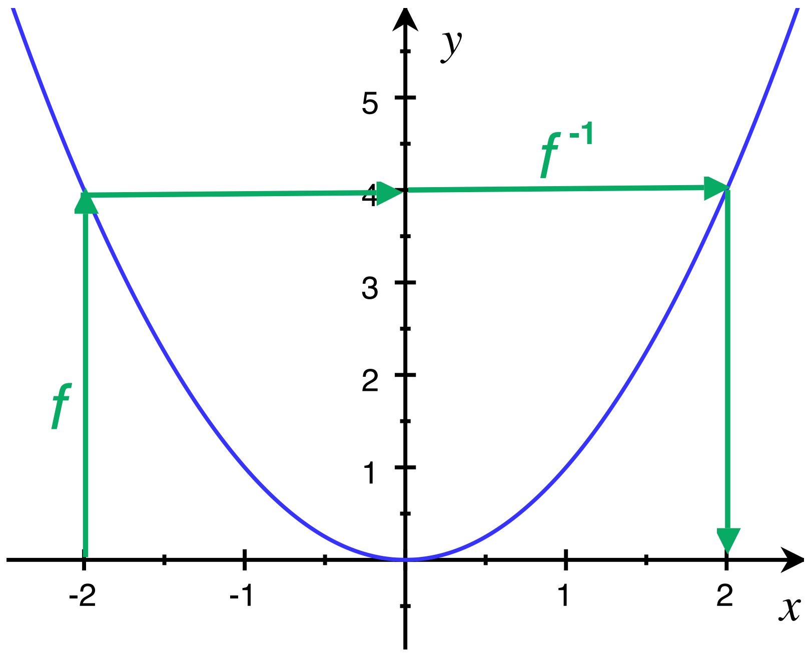


If $y = x^2$, then $x = |\sqrt{y}|$ or $-|\sqrt{y}|$. Which one???

If $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by

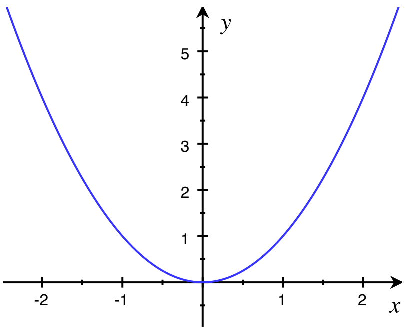
$$f(x) = x^2, \qquad \text{then} \qquad |\sqrt{x}| \text{ is not the inverse.}$$



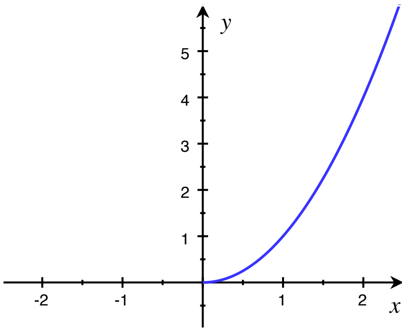
If $y = x^2$ and $x < 0$, then $x \neq |\sqrt{y}|$!

When is a function invertible?

A function f is one-to-one if no two inputs give the same output, that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
Example: over all real numbers, $f(x) = x^2$ *is not one-to-one*.
However, over non-negative real numbers, $f(x) = x^2$ *is one-to-one*.



no!



yes!

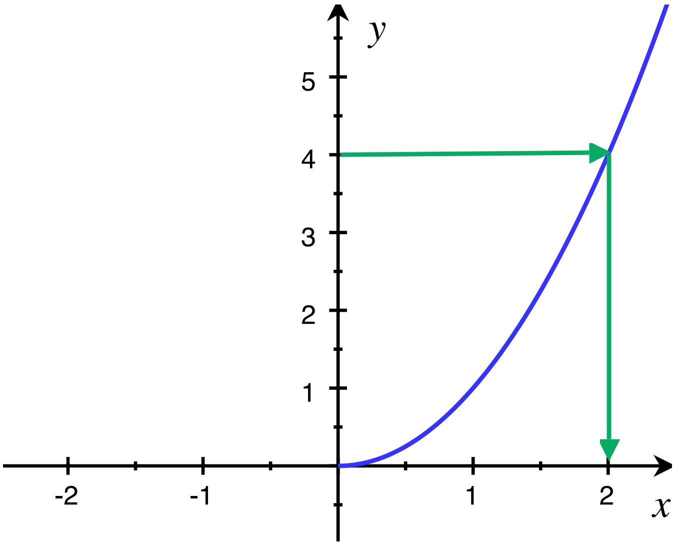
Horizontal line test: A function is one-to-one if and only if no horizontal line intersects the function's graph more than once.

Answer: A function is invertible if and only if it is one-to-one.

Graphing inverses

For a one-to-one function f, we have

$$f(x) = y \qquad \text{if and only if} \qquad x = f^{-1}(y)$$



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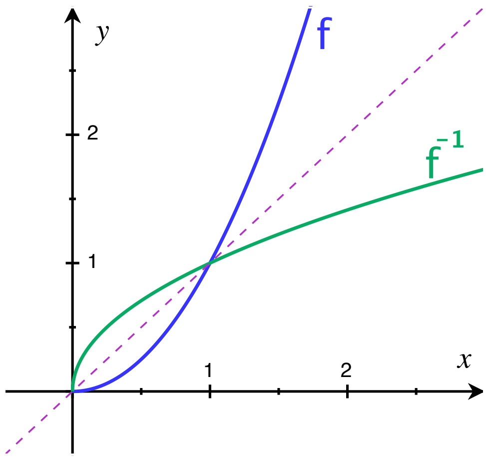


The graph of $y = f^{-1}(x)$ is the reflection of the graph of f over the line $y = x$ (i.e. swap the axes). Further,

the domain of f is the range of f^{-1},

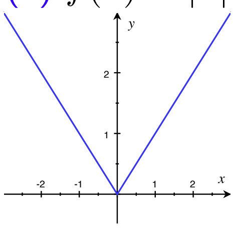
and

the range of f is the domain of f^{-1}.

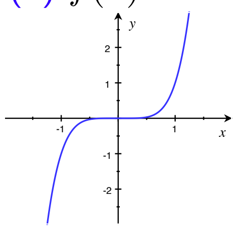
You try:

- For each of the following functions, (a) give the domain and range of f, and (b) decide if f is invertible.
- If f is invertible, then (c) sketch a graph of f^{-1}, (d) give the domain and range of f^{-1}, and (e) try to write a formula for f^{-1}.
- If f is not invertible over all of the real numbers, what is a restricted domain over which f is invertible? Over that restricted domain, do (c) and (d) from above.

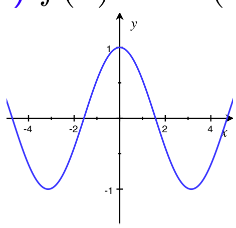
(1) $f(x) = |x|$



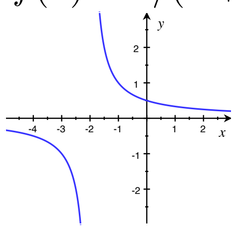
(2) $f(x) = x^5$



(3) $f(x) = \cos(x)$



(4) $f(x) = 1/(x+2)$



Calculating the inverse function algebraically

Given an invertible f, solve for f^{-1} by setting $f(y) = x$, and solving for $y = f^{-1}(x)$.

Example: Let $f(x) = 1/(x+2)$.

Set

$$x = f(y) = 1/(y+2).$$

Then

$$y + 2 = 1/x, \qquad \text{so that } \boxed{f^{-1}(x) = y = (1/x) - 2}.$$

Example: Let $f(x) = x^3 + 2$. (Check: is it invertible??)

Set

$$x = f(y) = y^3 + 2.$$

Then

$$y^3 = x - 2, \qquad \text{so that } \boxed{f^{-1}(x) = y = (x-2)^{1/3}}.$$

Note: As the book outlines, you can alternatively start with $f(x) = y$, solve for x, and then swap x and y at the end. You will get the same answer either way.

Checking your answer algebraically

Recall that f^{-1} is defined by

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Example: We calculated that if $f(x) = 1/(x+2)$, then $f^{-1}(x) = (1/x) - 2$. Let's check!

$$\begin{aligned} f(f^{-1}(x)) &= 1/((1/x) - 2 + 2) \\ &= 1/(1/x) = x \quad \checkmark \end{aligned}$$

and

$$\begin{aligned} f^{-1}(f(x)) &= (1/1/(x+2)) - 2 \\ &= x + 2 - 2 = x \quad \checkmark \end{aligned}$$

You try:

1. Check that if $f(x) = x^3 + 2$ then $f^{-1}(x) = (x-2)^{1/3}$ by calculating $f(f^{-1}(x))$ and $f^{-1}(f(x))$.
2. For the following functions, calculate $f^{-1}(x)$ and verify your answer as above. (a)$f(x) = 3/(x-1)$ (b)$f(x) = 5\sqrt{x-2}$

Derivatives of inverse functions

Note, if f is invertible and continuous, then f^{-1} is also continuous.

If $\boxed{f^{-1}(x) = y}$, then $\boxed{f(y) = x}$.

So we can use implicit differentiation to calculate $\frac{d}{dx} f^{-1}(x) = \frac{dy}{dx}$:

$$\frac{d}{dx} f(y) = \frac{d}{dx} x.$$

Now

$$\frac{d}{dx} f(y) = \underbrace{f'(y) * \frac{dy}{dx}}_{\text{chain rule!}} \qquad \text{and} \qquad \frac{d}{dx} x = 1.$$

So

$$f'(y) * \frac{dy}{dx} = 1.$$

Finally, solve for $\frac{d}{dx} f^{-1}(x) = \frac{dy}{dx}$:

$$\frac{d}{dx} f^{-1}(x) = \frac{dy}{dx} = 1/f'(y) = 1/f'(f^{-1}(x)).$$

Derivatives of inverse functions

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

Example: Let $f(x) = x^3 + 2$, so that $f^{-1}(x) = (x-2)^{1/3}$.
Let's calculate $\frac{d}{dx} f^{-1}(x)$ in two ways.
First, use the formula:

$$f'(x) = \frac{d}{dx}(x^3 + 2) = 3x^2,$$

so

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3((x-2)^{1/3})^2} = \frac{1}{3 * (x-2)^{2/3}}.$$

Now check by calculating directly:

$$\frac{d}{dx}(x-2)^{1/3} = (1/3) * (x-2)^{-2/3} = \frac{1}{3 * (x-2)^{2/3}}. \quad \checkmark$$

Derivatives of inverse functions at points

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

Example: Let $f(x) = 2x + \cos(x)$. Find $\frac{d}{dx}f^{-1}(x)\big|_{x=1}$.
One-to-one? Note that if a function is continuous and always increasing, then it must be one-to-one!

$$f'(x) = 2 - \sin(x) > 0 \qquad \text{so } f \text{ is one-to-one!}$$

Calculate the inverse? Why bother? Note that since $f(0) = 1$, we have $f^{-1}(1) = 0$.
Calculate the derivative:

$$\frac{d}{dx}f^{-1}(x)\Big|_{x=1} = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin(0)} = \boxed{\frac{1}{2}}.$$

You try:

For each of the following:

(a) Show that f is one-to-one.
(Show $f' \geq 0$ over the domain, and that f is never constant)

(b) Use the formula $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ to calculate $\frac{d}{dx}f^{-1}(x)\big|_{x=8}$.

(c) Calculate $f^{-1}(x)$, and state the domain and range of $f^{-1}(x)$.

(d) Calculate $\frac{d}{dx}f^{-1}(x)\big|_{x=8}$ directly using the formula from part (c). Check it against your answer for part (b).

(e) Sketch graphs of $f(x)$ and $f^{-1}(x)$ on the same axis. Then sketch a tangent line to $f^{-1}(x)$ at $x = 8$ and visually check that your answer to parts (b) and (d) roughly match the slope of this line.

(1) $f(x) = x^3$.	(2) $f(x) = 9 - x^2, 0 \leq x \leq 3$.