Math 201 lecture for Wednesday, Week 12

Orthogonal complements and projections

Definition. The *direct sum* of vector spaces U and W over a field F is the set

$$U \oplus W = \{(u, w) : u \in U \text{ and } w \in W\}$$

with scalar multiplication and vector addition defined by

$$\lambda(u, w) = (\lambda u, \lambda w) \quad \text{and} \quad (u, w) + (u', w') = (u + u', w + w'),$$

for all $u, u' \in U$, $w, w' \in W$, and $\lambda \in F$.

Proposition. Let U and W be subspaces of a vector space V over F such that: (i) the union of U and W spans V, and (ii) $U \cap W = \{0\}$. Then there is an isomorphism

$$\begin{aligned} U \oplus W &\to V \\ (u, w) &\mapsto u + w. \end{aligned}$$

Thus, every element of V has a unique expression of the form $u + w$ with $u \in U$ and $w \in W$.

Proof. Easy exercise. □

Remark. In the case of the Proposition, we says that V is the *internal direct sum* of U and W and abuse notation by simply writing $V = U \oplus W$. The direct sum as we first defined it is sometimes called the *external direct sum* of U and W.

For the rest of this lecture, let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Definition. Let $S \subseteq V$ be nonempty. The *orthogonal complement* of S is

$$S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}.$$

Exercise. Show that $S^\perp$ is a subspace of V.

Example. Consider $\mathbb{R}^3$ with the standard inner product, and let $S = \{(a, b, c)\}$. So S consists of the single vector $(a, b, c) \in \mathbb{R}^3$. Then

$$S^\perp = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (a, b, c) = 0\} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\},$$

a plane in $\mathbb{R}^3$ defined by the equation $ax + by + cz = 0$.

Proposition. Suppose $\dim V = n$ and $S = \{v_1, \ldots, v_k\}$ is an orthonormal subset of V.

(a) S can be extended to an orthonormal basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

(b) If $W = \operatorname{Span} S$, then $S' = \{v_{k+1}, \ldots, v_n\}$ is an orthonormal basis for $W^\perp$.

(c) If $W \subseteq V$ is any subspace, then

$$\dim W + \dim W^\perp = \dim V = n.$$

(d) If $W \subseteq V$ is any subspace, then $(W^\perp)^\perp = W$.

Proof. (a) To prove part (a), extend S to a basis $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ for V, then apply Gram-Schmidt.

(b) The set $S' = \{v_{k+1}, \ldots, v_n\}$ is linearly independent since it's a subset of a basis. Since $\{v_1, \ldots, v_n\}$ is orthonormal, and $W = \operatorname{Span}\{v_1, \ldots, v_k\}$, we have $S' \subseteq W^\perp$. Therefore, $\operatorname{Span} S' \subseteq W^\perp$. For the opposite inclusion, take $x \in W^\perp$. Then since $\{v_1, \ldots, v_n\}$ is orthonormal, we have

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle \, v_i = \sum_{i=k+1}^{n} \langle x, v_i \rangle \, v_i \in \operatorname{Span} S'.$$

(c) If $W \subseteq V$ is any subspace, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for W. Then apply parts (a) and (b).

(d) It's clear that $W \subseteq (W^\perp)^\perp$ since

$$(W^\perp)^\perp = \left\{ x \in V : \langle x, y \rangle = 0 \text{ for all } y \in W^\perp \right\}.$$

Then, by part (c),

$$\dim (W^\perp)^\perp = n - \dim W^\perp = \dim W.$$

Hence, $W = (W^\perp)^\perp$. □

Proposition. Let W be a finite-dimensional subspace of V. Then

$$V = W \oplus W^\perp.$$

In other words, for each $y \in V$, there exist unique $u \in W$ and $z \in W^\perp$ such that

$$y = u + z.$$

We define u to be the *orthogonal projection* of y onto W.

If $u_1, \ldots, u_k$ is an orthonormal basis for W, then

$$u = \sum_{i=1}^{k} \langle y, u_i \rangle \, u_i.$$

Proof. By Gram-Schmidt, there exists an orthonormal basis $u_1, \ldots, u_k$ for W. Define $u = \sum_{i=1}^{k} \langle y, u_i \rangle \, u_i$ and $z = y - u$. Then $u \in W$ and $y = u + z$. Further, $z \in W^\perp$ since for each $j = 1, \ldots, k$, we have

$$\begin{aligned}
\langle z, u_j \rangle &= \langle y - u, u_j \rangle \\
&= \langle y, u_j \rangle - \langle \textstyle\sum_{i=1}^{k} \langle y, u_i \rangle \, u_i, u_j \rangle \\
&= \langle y, u_j \rangle - \textstyle\sum_{i=1}^{k} \langle y, u_i \rangle \langle u_i, u_j \rangle \\
&= \langle y, u_j \rangle - \langle y, u_j \rangle \langle u_j, u_j \rangle
\end{aligned}$$

$$\begin{aligned} &= \langle y, u_j \rangle - \langle y, u_j \rangle \\ &= 0. \end{aligned}$$

For uniqueness, suppose there exist $u' \in W$ and $z' \in W^\perp$ such that

$$y = u + z = u' + z'.$$

Then $u - u' = z' - z \in W \cap W^\perp = \{0\}$. Thus, $u = u'$ and $z = z'$. (The reason $W \cap W^\perp = \{0\}$ is as follows: if $x \in W$, then we saw last time that $x = \sum_{i=1}^{k} \langle x, u_i \rangle \, u_i$. If it is also the case that $x \in W^\perp$, then $\langle x, u_i \rangle = 0$ for $i = 1, \ldots, k$ since each u_i is in W. Hence, $x = 0$.) □

Corollary. The orthogonal projection u of y onto W is the closest vector in W to y:

$$\|y - u\| \leq \|y - w\|$$

for all $w \in W$ with equality if and only if $w = u$.

Proof. Write $y = u + z$ with $u \in W$ and $z \in W^\perp$, and let $w \in W$. Then $u - w \in W$ and $y - u \in W^\perp$. So $u - w$ and $z = y - u$ are perpendicular. By the Pythagorean theorem,

$$\begin{aligned} \|y - w\|^2 &= \|(u + z) - w\|^2 \\ &= \|(u - w) + z\|^2 \\ &= \|(u - w)\|^2 + \|z\|^2 \\ &\geq \|z\|^2 \\ &= \|y - u\|^2. \end{aligned}$$

Equality occurs above if and only if $\|u - w\|^2 = 0$, i.e., if and only if $u = w$. □

Example. Let $V = \mathbb{R}^3$ with the standard inner product, and let's consider orthogonal projection onto the xy-plane. An orthonormal basis for the xy-plane is $\{e_1, e_2\}$. The projection of a point $u = (x, y, z) \in \mathbb{R}^3$ is given by

$$u = ((x, y, z) \cdot e_1)e_1 + ((x, y, z) \cdot e_2)e_2 = x\, e_1 + y\, e_2 = (x, y, 0).$$

The distance of (x, y, z) to the xy-plane is

$$\|(x, y, z) - u\| = \|(0, 0, z)\| = |z|.$$

Application. Consider the vector space V of integrable functions $f\colon [0, 2\pi] \to \mathbb{R}$ with inner product

$$\langle f, g \rangle := \frac{1}{\pi} \int_0^{2\pi} f(t)g(t)\, dt.$$

Thus, the distance between $f, g \in V$ is

$$\|f - g\| = \sqrt{\frac{1}{\pi} \int_0^{2\pi} (f(t) - g(t))^2\, dt},$$

which will be small if $f(t) \approx g(t)$ for $t \in [0, 2\pi]$.

One may check that $S_n := \left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx) \right\}$ is an orthonormal subset. Given any integrable $f \in V$, the orthogonal projection of f to the subspace spanned by S_n gives the best approximation of the function using sines and cosines of frequencies $\frac{j}{2\pi}$ for $j = 0, \ldots, n$. Write the projection of f to $\mathrm{Span}(S_n)$ as

$$\mathrm{proj}_{\mathrm{Span}(S_n)}(f)(x) = \alpha \cdot \frac{1}{\sqrt{2}} + \sum_{i=1}^{n} \beta_i \cos(ix) + \sum_{i=1}^{n} \gamma_i \sin(ix),$$

Since S_n is orthonormal, we may find the coefficients by taking inner products:

$$\alpha = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_0^{2\pi} \frac{f(t)}{\sqrt{2}}\, dt$$

$$\beta_i = \langle f, \cos(ix) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(ix)\, dt$$

$$\gamma_i = \langle f, \sin(ix) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(ix)\, dt.$$

For instance, consider the function $f(x) = x$ for $x \in [0, 2\pi]$. We find

$$\alpha = \frac{1}{\pi} \int_0^{2\pi} \frac{t}{\sqrt{2}}\, dt = \sqrt{2}\pi.$$

Integrating by parts, we find

$$\beta_i = \frac{1}{\pi} \int_0^{2\pi} t \cos(it)\, dt = \frac{1}{\pi} \left(\frac{t \sin(it)}{i} + \frac{\cos(it)}{i^2} \right|_0^{2\pi} = 0$$

and

$$\gamma_i = \frac{1}{\pi} \int_0^{2\pi} t \sin(it)\, dt = \frac{1}{\pi} \left(-\frac{t \cos(it)}{i} + \frac{\sin(it)}{i^2} \right|_0^{2\pi} = -\frac{2}{i}.$$

Thus,

$$\mathrm{proj}_{\mathrm{Span}(S_n)}(f)(x) = \sqrt{2}\pi \cdot \frac{1}{\sqrt{2}} - \sum_{i=1}^{n} \frac{2}{i\pi} \sin(ix) = \pi - \frac{2}{\pi} \sum_{i=1}^{n} \frac{\sin(ix)}{i}.$$

See the next page to compare the graph of f with the graphs of these projections for various n.

The plot of $\mathrm{proj}_{\mathrm{Span}(S_n)}(f)$ versus the plot of $f(x) = x$ for $n = 1, 2$, and 10.