Math 201 lecture for Wednesday, Week 11

Inner product spaces

We now add structure to a vector space allowing us to define length and angles.

Definition. Let V be a vector space over a field F where F is either $\mathbb{R}$ or $\mathbb{C}$. An *inner product* on V is a function

$$\begin{aligned} \langle\ ,\ \rangle\colon V\times V &\to F\\ (x,y)&\mapsto \langle x,y\rangle \end{aligned}$$

satisfying for all $x, y, z \in V$ and $c \in F$:

(a) linearity: $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ and $\langle cx,y\rangle = c\langle x,y\rangle$.

(b) conjugate symmetry: $\overline{\langle x,y\rangle} = \langle y,x\rangle$.

(c) positive-definiteness: $\langle x,x\rangle \in \mathbb{R}_{\geq 0}$, and $\langle x,x\rangle = 0$ iff $x = 0$.

Note. If $F = \mathbb{R}$, then an inner product is known as a *non-degenerate symmetric form.* If $F = \mathbb{C}$, an inner product is known as a *non-degenerate Hermitian form.*

Examples.

- The ordinary dot product on $\mathbb{R}^n$: Here, $V = \mathbb{R}^n$ and

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle = x\cdot y := \sum_{i=1}^n x_iy_i = x_1y_1+\cdots+x_ny_n.$$

  For example, in $\mathbb{R}^3$, we would have

$$\langle (1,2,3),(2,3,4)\rangle = 2+6+12 = 20 \quad\text{and}\quad \langle (1,2,3),(-2,1,0)\rangle = -2+2+0 = 0.$$

- The ordinary inner product on $\mathbb{C}^n$: Here, $V = \mathbb{C}^n$ and

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle = x\cdot \bar{y} := \sum_{i=1}^n x_i\bar{y_i} = x_1\bar{y_1}+\cdots+x_n\bar{y_n}.$$

  For example, in $\mathbb{C}^2$, we would have

$$\begin{aligned} \langle (1+i,1-i),(1+2i,4)\rangle &= (1+i)\overline{(1+2i)} + (1-i)\,\overline{4}\\ &= (1+i)(1-2i) + (1-i)\,4\\ &= (3-i)+(4-4i) = 7-5i. \end{aligned}$$

- Let $V = \mathcal{C}_{\mathbb{R}}([0,1]) = \{f\colon [0,1]\to\mathbb{R} : f \text{ is continuous}\}$, the vector space of $\mathbb{R}$-valued continuous functions on the interval $[0,1]$, and

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt.$$

To check positive-definiteness, note that if $f \neq 0$, then $f^2(t) > 0$ for t in some open interval in $[0, 1]$. Hence,

$$\langle f, f \rangle = \int_0^1 f^2(t)\, dt > 0.$$

- $V = \mathbb{R}^2$, and

  $$\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + 2x_1y_2 + 2x_2y_1 + 4x_2y_2.$$

  For positive-definiteness, we have

  $$\langle (x_1, x_2), (x_1, x_2) \rangle = 3x_1^2 + 4x_1x_2 + 4x_2^2.$$

  Complete the square:

  $$\begin{aligned} 3x_1^2 + 4x_1x_2 + 4x_2^2 &= 3\left(x_1^2 + \frac{4}{3}x_1x_2 + \frac{4}{3}x_2^2\right) \\ &= 3\left(\left(x_1 + \frac{2}{3}x_2\right)^2 - \frac{4}{9}x_2^2 + \frac{4}{3}x_2^2\right) \\ &= 3\left(\left(x_1 + \frac{2}{3}x_2\right)^2 + \frac{8}{9}x_2^2\right) \\ &\geq 0, \end{aligned}$$

  with equality if and only if $x_1 = x_2 = 0$.

- Let $F = \mathbb{R}$ or $\mathbb{C}$, and let $V = M_{m\times n}(F)$. For $A \in M_{m\times n}(F)$, define the *conjugate transpose* of A by

  $$A^* = \overline{A^t},$$

  where the overline means taking the conjugate of each entry of A. If A has only real entries, the $A^* = A^t$. Next, define the inner product,

  $$\langle A, B \rangle = \operatorname{tr}(B^*A) = \sum_{i=1}^{n} (B^*A)_{ii}.$$

  (Note: The special case $m = 1$ gives the usual inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$.) Proof of positive-definiteness is left as an exercise.

Proposition. Let $(V, \langle\, , \,\rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Then for all $x, y, z \in V$ and $c \in F$,

(a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.

(b) $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$.

(c) $\langle x, 0 \rangle = \langle 0, y \rangle = 0$.

(d) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$.

Proof. For part (a), notice that the definition of an inner product only guarantees sums on the left distribute. However, using properties of conjugation,

$$\begin{aligned}\langle x, y+z\rangle &= \overline{\langle y+z, x\rangle} \\ &= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} \\ &= \langle x, y\rangle + \langle x, z\rangle.\end{aligned}$$

Parts (b) and (c) are left as exercises. For part (d), $\langle x, y\rangle = \langle x, z\rangle$ for all x implies

$$\begin{aligned}0 = \langle x, y\rangle - \langle x, z\rangle &= \langle x, y\rangle + (-1)\langle x, z\rangle \\ &= \langle x, y\rangle + \overline{(-1)}\langle x, z\rangle \\ &= \langle x, y\rangle + \langle x, (-1)z\rangle \\ &= \langle x, y\rangle + \langle x, -z\rangle \\ &= \langle x, y-z\rangle\end{aligned}$$

for all x. In particular, let $x = y-z$ to get $\langle y-z, y-z\rangle = 0$. By positive-definiteness, $y-z = 0$. □