Math 201 lecture for Monday, Week 12

Gram-Schmidt

Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Definition. Let $S \subseteq V$. Then S is an *orthogonal* subset of V if $\langle u, v\rangle = 0$ for all $u, v \in S$ with $u \neq v$. If S is an orthogonal subset of V and $\|u\| = 1$ for all $u \in S$, then S is an *orthonormal* subset of V.

Examples.

- The standard basis $e_1, \dots, e_n$ for F^n is orthonormal with respect to the standard inner product on F^n.
- $\left\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\right\}$ is orthonormal with respect to the standard inner product on $\mathbb{R}^2$.

Proposition. Let $S = \{v_1, \dots, v_k\}$ be an orthogonal set of nonzero vectors in V, and let $y \in \operatorname{Span} S$. Then

$$y = \sum_{j=1}^{k} \frac{\langle y, v_j\rangle}{\langle v_j, v_j\rangle} v_j = \sum_{j=1}^{k} \frac{\langle y, v_j\rangle}{\|v_j\|^2} v_j.$$

Note that the coefficients are the components of y along each v_j.

Proof. Say $y = \sum_{i=1}^{k} a_i v_i$. Then for $j = 1, \dots, k$,

$$\langle y, v_j\rangle = \langle \textstyle\sum_{i=1}^{k} a_i v_i, v_j\rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j\rangle = a_j \langle v_j, v_j\rangle,$$

since $\langle v_i, v_j\rangle = 0$ for $i \neq j$. Hence,

$$a_j = \frac{\langle y, v_j\rangle}{\langle v_j, v_j\rangle} = \frac{\langle y, v_j\rangle}{\|v_j\|^2},$$

the component of y along v_j. $\square$

Corollary 1. If $S = \{v_1, \dots, v_k\}$ is orthonormal and $y \in \operatorname{Span} S$, then

$$y = \sum_{i=1}^{k} \langle y, v_j\rangle v_i.$$

Corollary 2. Is $S = \{v_1, \dots, v_k\}$ is an orthogonal set of nonzero vectors in V then S is linearly independent.

Proof. If $\sum_{i=1}^{k} a_i v_i = 0$, then for each $j = 1, \dots, k$,

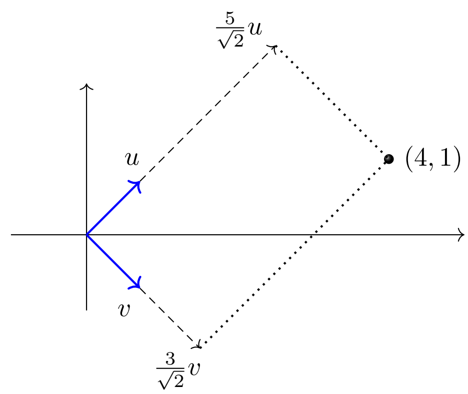
$$0 = \langle 0, v_j\rangle = \langle \textstyle\sum_{i=1}^{k} a_i v_i, v_j\rangle = a_j \langle v_j, v_j\rangle.$$

Since $v_j \neq 0$ and $\langle\ ,\ \rangle$ is positive-definite, we have $\langle v_j, v_j\rangle \neq 0$. Hence, $a_j = 0$ for $j = 1, \dots, k$. $\square$

Example. Consider $\mathbb{R}^2$ with the standard inner product, and let

$$u = \frac{1}{\sqrt{2}}(1,1) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(1,-1).$$

Then $\beta = \{u, v\}$ gives an orthonormal ordered basis for $\mathbb{R}^2$. What are the coordinates of $y = (4,1)$ with respect to that basis?



Answer:

$$\begin{aligned} y &= \langle y, u\rangle u + \langle y, v\rangle v \\ &= (4,1)\cdot\left(\frac{1}{\sqrt{2}}(1,1)\right)u + (4,1)\left(\frac{1}{\sqrt{2}}(1,-1)\right)v \\ &= \frac{5}{\sqrt{2}}\,u + \frac{3}{\sqrt{2}}\,v. \end{aligned}$$

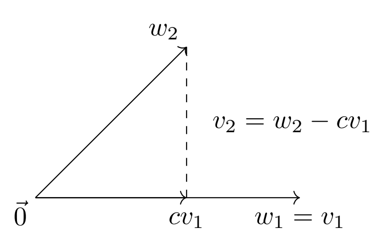
Check:

$$\frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,1)\right) + \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,-1)\right) = \frac{5}{2}\,(1,1) + \frac{3}{2}(1,-1) = (4,1).$$

Gram-Schmidt. Given vectors $w_1, w_2 \in V$, we'd like to compute orthogonal vectors v_1, v_2 such that

$$\text{Span}\,\{w_1, w_2\} = \text{Span}\,\{v_1, v_2\}\,.$$

To do that, let $v_1 = w_1$, then "straighten out" w_2 to create v_2:



The number c is the component of w_2 along v_1. Recall, c is determined by requiring v_2 and v_1 to be orthogonal:

$$0 = \langle v_2, v_1 \rangle = \langle w_2 - cv_1, v_1 \rangle = \langle w_2, v_2 \rangle - c\langle v_1, v_1 \rangle.$$

Therefore,

$$c = \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2}.$$

(We've assumed $v_1 \neq 0$.)

The following algorithm generalizes this idea:

Algorithm. (Gram-Schmidt orthogonalization)

INPUT: $S = \{w_1, \ldots, w_n\}$, a linearly independent subset of V.

Let

$$v_1 := w_1.$$

For $k = 2, 3, \ldots, n$, define v_k by starting with w_k, then subtracting off the components of w_k along the previously found v_i :

$$v_k := w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i.$$

OUTPUT: $S' = \{v_1, \ldots, v_n\}$ an orthogonal set with $\operatorname{Span} S' = \operatorname{Span} S$.

or

OUTPUT: $S'' = \left\{ \frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}$ an orthonormal set with $\operatorname{Span} S' = \operatorname{Span} S$.

Proof of validity of the algorithm. We prove this by induction on n. The case $n = 1$ is clear. Suppose the algorithm works for some $n \geq 1$, and let $S = \{w_1, \ldots, w_{n+1}\}$ be a linearly independent set. By induction, running the algorithm on the first n vectors in S produces orthogonal $v_1, \ldots, v_n$ with

$$\operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\}.$$

Running the algorithm further produces

$$v_{n+1} = w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i.$$

It cannot be that $v_{n+1} = 0$, since otherwise, the above equation we would say

$$w_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\},$$

contradicting the assumption of the linear independence of the w_i. So $v_{n+1} \neq 0$.

We now check that v_{n+1} is orthogonal to the previous v_i. For $j = 1, \ldots, n$, we have

$$\langle v_{n+1}, v_j \rangle = \left\langle w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i, v_j \right\rangle$$

$$= \langle w_{n+1}, v_j \rangle - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle$$

$$= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle$$

$$= \langle w_{n+1}, v_j \rangle - \langle w_{n+1}, v_j \rangle$$

$$= 0.$$

We have shown $\{v_1, \dots, v_{n+1}\}$ is an orthogonal set of vectors, and we would now like to show that its span is the span of $\{w_1, \dots, w_{n+1}\}$. First, since $\{v_1, \dots, v_{n+1}\}$ is orthogonal, it's linearly independent. Next, we have

$$\operatorname{Span}\{v_1, \dots, v_{n+1}\} \subseteq \operatorname{Span}\{v_1, \dots, v_n, w_{n+1}\} \subseteq \operatorname{Span}\{w_1, \dots, w_n, w_{n+1}\}.$$

Since $\operatorname{Span}\{v_1, \dots, v_{n+1}\}$ is an $(n+1)$-dimensional subspace of the $(n+1)$-dimensional space $\operatorname{Span}\{w_1, \dots, w_n, w_{n+1}\}$, these spaces must be equal. □

Corollary. Every nonzero finite-dimensional inner product space has an orthonormal basis.

Example. Let $V = \mathbb{R}_{\leq 1}[x]$, the space of polynomials of degree at most 1 with real coefficients and with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

Apply Gram-Schmidt to the basis $\{1, x\}$ to get an orthonormal basis. Note that 1 and x are not orthogonal:

$$\langle 1, x \rangle = \int_0^1 t\, dt = \frac{1}{2} \neq 0.$$

Gram-Schmidt: Start with $v_1 = 1$, then let

$$v_2 = x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= x - \frac{\langle x, 1 \rangle}{\|1\|^2} \cdot 1$$

$$= x - \frac{\int_0^1 t\, dt}{\int_0^1 dt} \cdot 1$$

$$= x - \frac{1}{2}.$$

Check orthogonality:

$$\langle 1, x - 1/2 \rangle = \int_0^1 (t - 1/2)\, dt = 0.$$

Now scale $v_1 = 1$ and $v_2 = x - 1/2$ to create an orthonormal basis:

$$\|v_1\| = \sqrt{\int_0^1 dt} = 1$$

$$
\begin{aligned}
\|v_2\| &= \sqrt{\langle x - 1/2, x - 1/2 \rangle} \\
&= \sqrt{\int_0^1 (t - 1/2)^2 \, dt} \\
&= \sqrt{\int_0^1 (t^2 - t + 1/4) \, dt} \\
&= \sqrt{1/12}.
\end{aligned}
$$

So an orthonormal basis for V is

$$
\left\{1, \sqrt{12}(x - 1/2)\right\}.
$$