Lecture 23:

Orthogonal complement
Orthogonal projection

Let $F = \mathbb{R}$ or $\mathbb{C}$, and let V be a vector space over F. Let $\langle , \rangle : V \times V \to F$ be an inner product (linear in the first coordinate, conjugate symmetric, and positive definite).

Review: We say a set $S \subseteq V$ is **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{u} \neq \mathbf{v}$ in S. We say S is **orthonormal** if, additionally, $\langle \mathbf{u}, \mathbf{u} \rangle = 1$ for all $\mathbf{u} \in S$.

Prop. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$ is orthogonal and $\mathbf{y} \in FS$, then

$$\mathbf{y} = \sum_{i=1}^{k} \frac{\langle \mathbf{y}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \mathbf{v}_i.$$

In particular, S is linearly independent.

Gram-Schmidt orthogonalization is an algorithm (using this proposition) for turning an independent set into an orthogonal set with the same span.

The **direct sum** of vector spaces U and W over a field F is the set

$$U \oplus W = \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$$

with scalar multiplication and vector addition defined by

$$\lambda(\mathbf{u}, \mathbf{w}) = (\lambda \mathbf{u}, \lambda \mathbf{w}) \quad \text{and} \quad (\mathbf{u}, \mathbf{w}) + (\mathbf{u}', \mathbf{w}') = (\mathbf{u} + \mathbf{u}', \mathbf{w} + \mathbf{w}'),$$

for all $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{w}, \mathbf{w}' \in W$, and $\lambda \in F$.

[There is a *very subtle* difference between Cartesian product and direct sum, which will only come up once you get into *infinite* products/sums. But for now, qualitatively, this combination of vectors spaces acts more like addition than multiplication.]

Example: If $U = \mathbb{C}^3$ and $W = \mathbb{C}^5$, then

$$\begin{aligned} U \oplus W &= \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U, \mathbf{w} \in W\} \\ &= \{\big((a_1, a_2, a_3), (b_1, b_2, b_3, b_4, b_5)\big) \mid a_i \in \mathbb{C}, b_i \in \mathbb{C}\}. \end{aligned}$$

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for all $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{w}, \mathbf{w}' \in W$, and $\lambda \in F$.

Some facts:

- If $\mathcal{A}$ is a basis of U and $\mathcal{B}$ is a basis of W, then
  $$\hat{\mathcal{A}} \sqcup \hat{\mathcal{B}} \text{ is a basis of } U \oplus W,$$
  where
  $$\hat{\mathcal{A}} = \{(\mathbf{a}, \mathbf{0}) \mid \mathbf{a} \in \mathcal{A}\} \quad \text{and} \quad \hat{\mathcal{B}} = \{(\mathbf{0}, \mathbf{b}) \mid \mathbf{b} \in \mathcal{B}\}.$$
  In particular, if $\dim(U) = k$ and $\dim(W) = \ell$, then $\dim(U \oplus W) = k + \ell$.
- Let
  $$\hat{U} = F\hat{\mathcal{A}} = \{(\mathbf{u}, \mathbf{0}) \mid \mathbf{u} \in U\} \quad \text{and} \quad \hat{W} = F\hat{\mathcal{B}} = \{(\mathbf{0}, \mathbf{w}) \mid \mathbf{w} \in W\}.$$
  Then
  $$\begin{aligned} U &\to \hat{U} \\ \mathbf{u} &\mapsto (\mathbf{u}, \mathbf{0}) \end{aligned} \quad \text{and} \quad \begin{aligned} W &\to \hat{W} \\ \mathbf{w} &\mapsto (\mathbf{0}, \mathbf{w}) \end{aligned}$$
  are isomorphisms.

Internal versus external direct sums

Prop. Let U and W be subspaces of a vector space V over F such that

(i) the union of U and W spans V, and (ii) $U \cap W = \{0\}$.

Then there is an isomorphism

$$\begin{aligned} f : U \oplus W &\to V \\ (\mathbf{u}, \mathbf{w}) &\mapsto \mathbf{u} + \mathbf{w}. \end{aligned}$$

Thus, every element of V has a unique expression of the form $\mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Pf. First, f is linear: for $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{w}, \mathbf{w}' \in W$, and $\lambda \in F$, we have

$$\begin{aligned} (\mathbf{u}, \mathbf{w}) + \lambda(\mathbf{u}', \mathbf{w}') &= (\mathbf{u} + \lambda\mathbf{u}', \mathbf{w} + \lambda\mathbf{w}') \\ &\overset{f}{\longmapsto} \mathbf{u} + \lambda\mathbf{u}' + \mathbf{w} + \lambda\mathbf{w}' = (\mathbf{u} + \mathbf{w}) + \lambda(\mathbf{u}' + \mathbf{w}'). \end{aligned}$$

Next, f is surjective by definition. Finally, to see that f is injective, we compute its kernel:

$$\text{if } \mathbf{u} + \mathbf{w} = 0 \quad \text{then} \quad \mathbf{u} = -\mathbf{w} \in W$$

because W is closed under scaling. Hence $\mathbf{u}, \mathbf{w} \in U \cap W = 0$. □

We call $U \oplus W$ an **external direct sum** (two unrelated-ish separate spaces make a new space). If $U, W \subseteq V$ satisfy the above conditions, we call $U + W$ an **internal direct sum** (they're subspaces internal to a common space).

Example. Let $Y \in M_5(\mathbb{C})$ be a matrix with characteristic equation

$$p_Y(x) = (x-1)^2(x+4)^3.$$

Then

$$\dim(V^1(Y)) = 2 \qquad \dim(V^{-4}(Y)) = 3 \qquad \text{and} \qquad V^1(Y) \cap V^{-4}(Y) = 0.$$

(Recall $V^\lambda(Y)$ is the generalized eigenspace of eigenvalue λ.)

Therefore

$$V^1(Y) + V^{-4}(Y) \cong V^1(Y) \oplus V^{-4}(Y).$$

Moreover, since

$$\dim(V^1(Y) \oplus V^{-4}(Y)) = \dim(V^1(Y)) + \dim(V^{-4}(Y)) = 2 + 3 = 5 = \dim(V),$$

we have $V^1(Y) + V^{-4}(Y) = V$, and hence $V \cong \dim(V^1(Y) \oplus V^{-4}(Y))$.

Orthogonal complement

$(V, \langle , \rangle)$ and IPS$/F = \mathbb{R}$ or $\mathbb{C}$

Let $S \subseteq V$ be nonempty. The **orthogonal complement** of S is

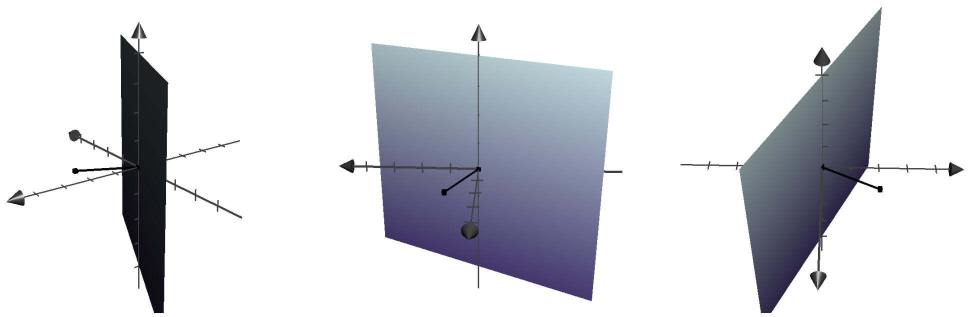
$$S^\perp = \{\mathbf{x} \in V \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in S\}.$$

Example. Let $V = \mathbb{R}^3$ and $\langle , \rangle$ dot product. Let $S = \{(1, 2, 0)\}$. Then

$$0 = (a, b, c) \cdot (1, 2, 0) = a + 2b.$$

So

$$\begin{aligned} S^\perp &= \{(a, b, c) \in \mathbb{R}^3 \mid (a, b, c) \cdot (1, 2, 0) = 0\} \\ &= \{(-2b, b, c) \mid b, c \in \mathbb{R}\} \end{aligned}$$



Note: If $S' = FS$, then $(S')^\perp = S^\perp$.

Orthogonal projection

$(V, \langle , \rangle)$ and IPS$/F = \mathbb{R}$ or $\mathbb{C}$

$S^\perp = \{\mathbf{x} \in V \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in S\}$.

Prop. $S^\perp$ is a subspace of V.

Pf. Use the subspace criterion...

1. $S^\perp \neq \emptyset$:
2. For $\mathbf{x}, \mathbf{x}' \in S^\perp$ and $\lambda \in F$, $\lambda\mathbf{x} + \mathbf{x}' \in S^\perp$:

Theorem. Let W be a **finite-dimensional** subspace of V. Then

$$V \cong W \oplus W^\perp.$$

Thus, for each $\mathbf{y} \in V$, there exist unique $\mathbf{u} \in W$ and $\mathbf{z} \in W^\perp$ such that

$$\mathbf{y} = \mathbf{u} + \mathbf{z}.$$

Idea: If $\mathbf{v} \in W \cap W^\perp$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. So $\mathbf{v} = \mathbf{0}$.

It remains to show that $W + W^\perp = V$.

The vector $\mathbf{u} \in W$ is $\mathrm{proj}_W(\mathbf{y})$, the **orthogonal projection** of $\mathbf{y}$ onto W (we'll see below how to compute $\mathbf{u}$).

Fact: $\mathrm{proj}_W(\mathbf{y})$ is the unique closest vector to $\mathbf{y}$ that is in W.

(Think about the case when $\dim(W) = 1$. Pf in a moment.)

Proposition. Suppose $\dim(V) = n$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal subset of V (orthogonal and all norm 1).

(a) S can be extended to an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for V.

Pf. S can be extended to an (ordered) basis $\mathcal{A}$. Then perform Gram-Schmidt orthogonalization on $\mathcal{A}$ (to get $\mathcal{B}$). If you process S first, this process will preserve S.

(b) If $W = FS$, then $S' = \{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for $W^\perp$.

Pf. By Gram-Schmidt, $S' \subseteq W^\perp$ so that $FS' \subseteq W^\perp$. Let $\mathbf{x} \in W^\perp$. Since $\mathcal{B}$ is an orthonormal basis of V,

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i = \mathbf{0} + \sum_{i=k+1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i \in FS'.$$

Thus $W^\perp = FS'$. So since S' is independent, it's a basis.

(c) If $W \subseteq V$ is any subspace, then

$$\dim W + \dim W^\perp = \dim V = n.$$

(d) If $W \subseteq V$ is any subspace, then $(W^\perp)^\perp = W$.

Pf. Use conjugate symmetry.

Orthogonal projection

Back to our theorem:

Theorem. Let W be a **finite-dimensional** subspace of V. Then

$$V \cong W \oplus W^\perp.$$

Thus, for each $\mathbf{y} \in V$, there exist unique $\mathbf{u} \in W$ and $\mathbf{z} \in W^\perp$ such that

$$\mathbf{y} = \mathbf{u} + \mathbf{z}.$$

We call $\mathbf{u} = \mathrm{proj}_W(\mathbf{y})$ the **orthogonal projection** of $\mathbf{y}$ onto W.

Pf. If $\mathcal{A} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonm. basis for W, let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ be an orthonm. basis extending $\mathcal{A}$. We saw $\mathcal{A}' = \mathcal{B} - \mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$ is an orthonm. basis of $W^\perp$. In particular, $W + W^\perp = F\mathcal{A} + F\mathcal{A}' = V$, so since $W \cap W^\perp = 0$, we have $V \cong W \oplus W^\perp$.

Then to compute $\mathbf{u} = \mathrm{proj}_W(\mathbf{y})$: we have

$$\mathbf{y} = \sum_{\mathbf{b} \in \mathcal{B}} \langle \mathbf{y}, \mathbf{b} \rangle \mathbf{b} = \underbrace{\sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{u}_i \rangle \mathbf{u}_i}_{\mathbf{u} \in W} + \underbrace{\sum_{j=1}^{n-k} \langle \mathbf{y}, \mathbf{v}_j \rangle \mathbf{v}_j}_{\mathbf{z} \in W^\perp}.$$

So

$$\mathrm{proj}_W(\mathbf{y}) = \sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{u}_i \rangle \mathbf{u}_i.$$

Corollary. The orthogonal projection $\mathrm{proj}_W(\mathbf{y})$ of $\mathbf{y}$ onto W is the closest vector in W to $\mathbf{y}$:

$$\|\mathbf{y} - \mathrm{proj}_W(\mathbf{y})\| \le \|\mathbf{y} - \mathbf{w}\|$$

for all $\mathbf{w} \in W$, with equality if and only if $\mathbf{w} = \mathrm{proj}_W(\mathbf{y})$.

Pf. Write $\mathbf{y} = \mathbf{u} + \mathbf{z}$ with $\mathbf{u} \in W$ and $\mathbf{z} \in W^\perp$ (so that $\mathbf{u} = \mathrm{proj}_W(\mathbf{y})$), and let $\mathbf{w} \in W$. So

$$\mathbf{u} - \mathbf{w} \in W \quad \text{and} \quad \mathbf{y} - \mathbf{u} \in W^\perp$$

Thus $\mathbf{u} - \mathbf{w}$ and $\mathbf{z} = \mathbf{y} - \mathbf{u}$ are perpendicular!

The result now follows from the Pythagorean theorem:

$$\|\mathbf{y} - \mathbf{w}\|^2 = \|(\mathbf{u} + \mathbf{z}) - \mathbf{w}\|^2 =$$

Applications (presentations!)

Application 1 (kind of). 3D graphics (project a 3-dimensional object onto a plane as cameral angles change).

Application 2: Fourier Series (presentations next week)
Let V be the vector space of integrable functions $[0, 2\pi] \to \mathbb{R}$ with inner product

$$\langle f, g \rangle := \frac{1}{\pi} \int_0^{2\pi} f(t)g(t)\, dt.$$

The distance between $f, g \in V$ is

$$\|f - g\| := \frac{1}{\pi} \int_0^{2\pi} (f(t) - g(t))^2\, dt.$$

One orthonomal set with respect to this product is

$$S_n := \left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(nx), \sin(nx) \right\}.$$

Problem: Approximate $f \in V$ with an element in $F(S_n)$.

You try: Let $V = \mathbb{C}^4$ and let $\langle , \rangle$ be the conjugate dot product ($\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \overline{\mathbf{v}}$). Let

$$Y = \begin{pmatrix} 6 & 6 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -5 & -4 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \in M_4(\mathbb{C}).$$

(a) We have $p_Y(x) = (x+i)(x-i)(x-5)^2$. Compute $W = V_5(Y)$ (the $\lambda = 5$ eigenspace of Y), and give a basis of W.

(b) Use Gram-Schmidt to compute an orthogonal basis $\mathcal{A}$ of W. Check your answer (compute the inner products).

(c) Normalize the basis $\mathcal{A}$ in the previous part to get an orthonormal basis $\mathcal{A}'$ of W.

(d) Use the previous part to project each of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$ onto W. [Reality check: Verify that each of your 4 answers $\mathbf{u}_i = \mathrm{proj}_W(\mathbf{e}_i)$ are, indeed, eigenvectors of Y by computing $Y\mathbf{u}_i$.]

(e) Extend your basis $\mathcal{A}'$ to a basis $\mathcal{B}$ of V (try adding in $\mathbf{e}_i$'s, one at a time, that aren't already in the span). Use Gram-Schmidt to compute an orthogonal basis $\mathcal{B}'$ of V that contains $\mathcal{A}'$.

(f) Are the remaining vectors ($\mathbf{v} \in \mathcal{B}' - \mathcal{A}'$) eigenvectors?

(Is there any good reason to expect that they are?)