Lecture 21:

Inner products
Inner produce spaces
Orthogonal projection
Length and distance

Let $F = \mathbb{R}$ or $\mathbb{C}$.

Note: $V = \mathbb{C}$ is a vector space over $\mathbb{R}$, with basis $B = \{1, i\}$: $\text{Rep}_B(x+iy) = \begin{pmatrix} x \\ y \end{pmatrix}$

Complex conjugation is a linear function!

$\text{Rep}_B^B(\overline{*}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ So B is an eigenbasis.

$\overline{\alpha} = \alpha$ means $\alpha \in V_1(\overline{*}) = \mathbb{C}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$.

Inner products

Let V be a vector space over a field F where $F = \mathbb{R}$ or $\mathbb{C}$. Recall that the **complex conjugate** is

$$\mathbb{C} \to \mathbb{C} \quad \text{defined by} \quad \overline{x + iy} = x - iy,$$

for $x, y \in \mathbb{R}$. In particular, for $\alpha \in \mathbb{C}$,

$$\overline{\alpha} = \alpha \quad \text{if and only if} \quad \alpha \in \mathbb{R}.$$

An **inner product** on V is a function

$$\begin{array}{rcl} \langle\ ,\ \rangle \colon V \times V & \to & F \\ (\mathbf{u}, \mathbf{v}) & \mapsto & \langle \mathbf{u}, \mathbf{v} \rangle \end{array}$$

satisfying the following...

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in F$,

1. **linearity** (in the first coordinate): $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$;
2. **conjugate symmetry:** $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$; and
3. **positive-definiteness:** $\langle \mathbf{u}, \mathbf{u} \rangle \in \mathbb{R}_{\geq 0}$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Favorite examples:

1. The standard inner product on $\mathbb{R}^n$ is dot product:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n.$$

2. The standard inner product on $\mathbb{C}^n$ is the conjugate dot product:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := \mathbf{x} \cdot \overline{\mathbf{y}} = \sum_{i=1}^{n} x_i \overline{y_i} = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}.$$

3. Let $V = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ with $F = \mathbb{R}$. Then

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt \qquad \text{is an inner product.}$$

You try: Check that these are inner products (see next slide for details).

1. Let $\langle , \rangle$ be dot product on $\mathbb{R}^n$.
   (a) Briefly check that $\langle , \rangle$ is linear in the first coordinate and symmetric (conjugate symmetry is just symmetry in $\mathbb{R}$). [We've already done the necessary *proofs*]
   (b) Use the geometric interpretation to briefly check that $\langle , \rangle$ is positive definite.
2. Let $\langle , \rangle$ be conjugate dot product on $\mathbb{C}^n$.
   (a) Compute (i) $\langle (1+i, 2-3i), (5i, 2-3i) \rangle$, (ii) $\langle (1,2), (3,-1) \rangle$, and (iii) $\langle (a+ib, c+id), (a+ib, c+id) \rangle$ (for any $a, b, c, d \in \mathbb{R}$).
   (b) Briefly check that $\langle , \rangle$ is linear in the first coordinate.
   (c) Why is $\langle , \rangle$ *conjugate* symmetric?
   (d) Check algebraically that $\langle , \rangle$ is positive definite.
   [*Hint.* See (a)(iii): for $\alpha = x + iy \in \mathbb{C}$, what can you say about $\alpha\overline{\alpha}$?]
3. *Time permitting:* Consider example 3 above, with
   $V = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ and $\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$.
   (a) Compute $\langle f, g \rangle$ when (i) $f(x) = x^2$ and $g(x) = 3x + 2$,
   (ii) $f(x) = (x+2)e^x$ and $g(x) = \frac{3}{x+2}$, and
   (iii) $f(x)$ is any continuous function and $g(x) = 0$.
   (b) Check that (i) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$, (ii) $\langle cf, g \rangle = c\langle f, g \rangle$, and (iii) $\langle f, g \rangle = \langle g, f \rangle$ for all $f, g, h \in V$ and $c \in \mathbb{R}$.
   (c) Check that if ζ is the zero function, then $\langle \zeta, \zeta \rangle = 0$.
   (d) Show that if $f \neq \zeta$, then $\langle f, f \rangle \in \mathbb{R}_{>0}$.
   [*Hint.* Note that $(f(t))^2 \geq 0$ for all $t \in [0,1]$; and $(f(t))^2 > 0$ whenever $f(t) \neq 0$. And if $f \neq \zeta$, then $f \neq 0$ for some open interval $(a,b) \subseteq [0,1]$ (f is continuous).]

An inner product is a function $\langle\ ,\ \rangle \colon V \times V \to F$ that is linear (in the first coordinate), conjugate symmetric, and positive definite.

A vector space V together with an inner product $\langle , \rangle$ is called an **inner product space (IPS)**.

Proposition 1. Let $(V, \langle , \rangle)$ be an inner product space (IPS) over $F = \mathbb{R}$ or $\mathbb{C}$. Then for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in F$, we have the following.

(a) **conjugate linear** in the second coordinate:
$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ and $\langle \mathbf{u}, c\mathbf{v} \rangle = \overline{c} \langle \mathbf{u}, \mathbf{v} \rangle$;

(b) $\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle = 0$; and

(c) **nondegenerate:** if $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u} \in V$, then $\mathbf{v} = \mathbf{w}$.

To prove these, recall that complex conjugation is

- a field homomorphism (meaning it preserves field structure of $\mathbb{C}$):
$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta} \quad \text{and} \quad \overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}; \qquad \text{and}$$
- an involution (meaning that it is its own inverse):
$$\overline{(\overline{\alpha})} = \alpha.$$

Recall that the **standard norm on $\mathbb{C}$** itself is given by
$$|x + iy|^2 = (x + iy)\overline{(x + iy)} = x^2 + y^2.$$

So $|x+iy| = \sqrt{x^2+y^2}$

Note: When $F = \mathbb{R}$,

- "conjugate symmetric" is just symmetric;
- "conjugate linear" is just linear: in this case we say $\langle , \rangle$ is **bilinear**.

Another name for an inner product on a real vector space is a symmetric, bilinear, positive definite **form**.

An inner product on a complex vector space is also called a **Hermitian form**.

"form" loosely means a function

$$\underbrace{V \times \cdots \times V}_{k} \longrightarrow F$$

Norms

Still: Let $(V, \langle , \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

We can use $\langle , \rangle$ to define some geometric notions on V as follows.

The induced **norm** or **length** of $\mathbf{v} \in V$ is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \in \mathbb{R}_{\geq 0}.$$

(*Must check:* positive when $\mathbf{v} \neq \mathbf{0}$, homogeneous ($\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$), and satisfies the triangle inequality ($\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.)

Examples: For $V = \mathbb{R}^n$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, we have

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

For $V = \mathbb{C}^n$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \overline{\mathbf{v}}$, then

$$\|\mathbf{v}\| = \sqrt{v_1 \overline{v_1} + \cdots + v_n \overline{v_n}} = \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

[*Note:* this is just the standard norm on $\mathbb{C}$ with $n = 1$!]

We say $\mathbf{u} \in V$ is a **unit vector** if $\|\mathbf{u}\| = 1$; equivalently, if $\langle \mathbf{u}, \mathbf{u} \rangle = 1$.

Triangle inequality:

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle && \text{by definition} \\ &= \langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\ &= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle && \text{using linear/conj linear.} \end{aligned}$$

Piece-by-piece, we have

$$\begin{aligned} \langle \mathbf{u},\mathbf{u}\rangle &= \|\mathbf{u}\|^2 \\ \langle \mathbf{v},\mathbf{v}\rangle &= \|\mathbf{v}\|^2, \quad \text{and} \\ \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle &= \underbrace{\langle \mathbf{u},\mathbf{v}\rangle}_{\in\mathbb{C}} + \underbrace{\overline{\langle \mathbf{u},\mathbf{v}\rangle}}_{\in\mathbb{C}} \qquad [(x+iy)+(x-iy)=2x] \\ &= 2\,\mathrm{Re}(\langle \mathbf{u},\mathbf{v}\rangle). \end{aligned}$$

Let $|x+iy| = \sqrt{x^2+y^2}$ (the usual norm on $\mathbb{C}$). So

$$\mathrm{Re}(x+iy) = x \le \sqrt{x^2+y^2} = |x+iy|.$$

Therefore

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\langle \mathbf{u},\mathbf{v}\rangle|$$

Goal: understand $|\langle \mathbf{u},\mathbf{v}\rangle|$.

We say $\mathbf{u},\mathbf{v}\in V$ are **orthogonal** or **perpendicular** if $\langle \mathbf{u},\mathbf{v}\rangle = 0$.

Shorthand: $\mathbf{u}\perp\mathbf{v}$.

Example: For $V=\mathbb{R}^n$ and $\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v}$, we saw

$$\langle \mathbf{u},\mathbf{v}\rangle = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta), \qquad \text{where } \theta = \mathbf{u}\angle\mathbf{v}.$$

So $\langle \mathbf{u},\mathbf{v}\rangle = 0$ exactly when $\theta = \pm\pi/2$, or one of $\mathbf{u}$ or $\mathbf{v}$ is $\mathbf{0}$.

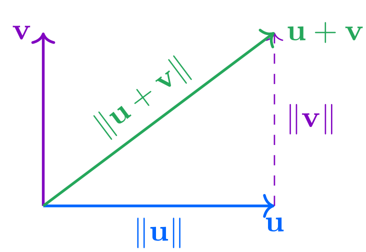
Example: As a vector space over $\mathbb{R}$, $\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$ via

$$f:(x_1+iy_1,\dots,x_n+iy_n)\to(x_1,y_1,\dots,x_n,y_n).$$

This isomorphism preserves norms: the norm of $\mathbf{v}\in\mathbb{C}^n$ using the conjugate dot product is the norm of $f(\mathbf{v})$ using the regular dot product.

Proposition 2. (Pythagorean theorem) Let $(V,\langle,\rangle)$ be an inner product space over $F=\mathbb{R}$ or $\mathbb{C}$. If $\mathbf{u},\mathbf{v}\in V$ are **perpendicular**, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$



Pf. $|\langle \mathbf{u},\mathbf{v}\rangle| = 0$ above.

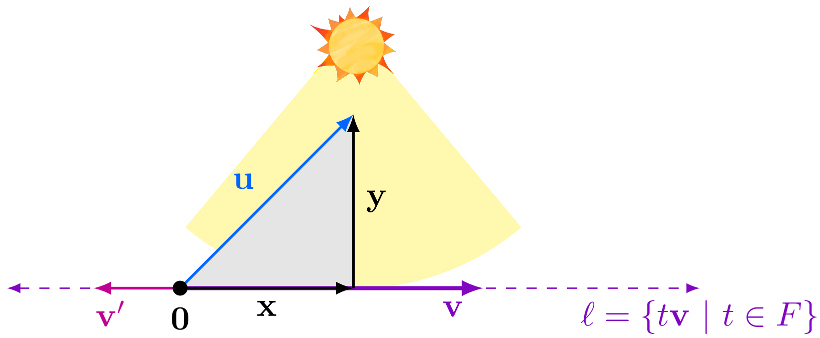
Components and projections

Still: Let $(V, \langle,\rangle)$ be an IPS/$F = \mathbb{R}$ or $\mathbb{C}$.

Let $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq \mathbf{0}$.

Goal: Find $\mathbf{x}, \mathbf{y} \in V$ such that

$\mathbf{v}$ is parallel to $\mathbf{x}$ and perp. to $\mathbf{y}$ and $\mathbf{x} + \mathbf{y} = \mathbf{u}$:



Idea: The vector $\mathbf{x}$ is the "shadow" of $\mathbf{u}$ along the line generated by $\mathbf{v}$. Our answer shouldn't depend on *which* representative vector we picked along ℓ!

We call $\mathbf{x}$ the **orthogonal projection** of $\mathbf{u}$ to $\mathbf{v}$, denoted

$$\boxed{\mathbf{x} = \operatorname{proj}_{\mathbf{v}}(\mathbf{u}).}$$

To compute: Note that $\mathbf{x} = c\mathbf{v}$ for some $c \in F$ and $\mathbf{y} = \mathbf{u} - \mathbf{x} = \mathbf{u} - c\mathbf{v}$. So

$$0 = \langle \mathbf{y}, \mathbf{v} \rangle = \langle \mathbf{u} - c\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - c\langle \mathbf{v}, \mathbf{v} \rangle. \qquad \text{So } c = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

The **component** of $\mathbf{u}$ along $\mathbf{v}$ is the scalar

$$c_{\mathbf{u},\mathbf{v}} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}.$$

The **orthogonal projection** of $\mathbf{u}$ to $\mathbf{v}$ is the vector $\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = c_{\mathbf{u},\mathbf{v}}\mathbf{v}$.

Let $(V, \langle , \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Proposition 3. Let $\mathbf{u}, \mathbf{v} \in V$ and $c \in F$. Then

(a) $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$. *Pf.* $\|c\mathbf{u}\| = \sqrt{\langle c\mathbf{u}, c\mathbf{u}\rangle} = \sqrt{c\bar{c}\langle \mathbf{u}, \mathbf{u}\rangle} = \cdots$

(b) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$. *Pf.* (exercise)

(c) **Cauchy-Schwarz inequality:** $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.
Pf. If $\mathbf{v} = \mathbf{0}$, we're done; so consider $\mathbf{v} \neq \mathbf{0}$. We can apply the Pythagorean Theorem to

$$\mathbf{x} = c_{\mathbf{u},\mathbf{v}}\mathbf{v} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v} \quad \text{and} \quad \mathbf{y} = \mathbf{u} - \mathbf{x} = \mathbf{u} - c_{\mathbf{u},\mathbf{v}}\mathbf{v},$$

as above! Namely,

$$\|\mathbf{u}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \geq \|\mathbf{x}\|^2$$

But

$$\|\mathbf{x}\| = \|c_{\mathbf{u},\mathbf{v}}\mathbf{v}\| = |c_{\mathbf{u},\mathbf{v}}|\|\mathbf{v}\| = \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|}{\|\mathbf{v}\|^2}\|\mathbf{v}\| = \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|}{\|\mathbf{v}\|},$$

since $\|\mathbf{v}\|^2 \in \mathbb{R}_{>0}$. Thus $\|\mathbf{u}\| \geq \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|}{\|\mathbf{v}\|}$.

(d) **Triangle inequality:** $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
Pf. Use (c) in our calculation above.

Distance and metric spaces

Still: Let $(V, \langle , \rangle)$ be an IPS$/F = \mathbb{R}$ or $\mathbb{C}$.

Define the distance between points $\mathbf{x}, \mathbf{y} \in V$ as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Proposition. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$,

1. Symmetry: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
2. Positive-definiteness: $d(\mathbf{x}, \mathbf{y}) \geq 0$, and $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
3. Triangle inequality: $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

Namely, an inner product space is also a **metric space**.

You try.

1. Let $\mathbf{u} = (1, 0, -2)$. For each of the following $\mathbf{v}$, compute $\text{proj}_{\mathbf{v}}(\mathbf{u})$.
   (a) $\mathbf{v} = (3, -1, 4)$;
   (b) $\mathbf{v} = (1, 0, 0)$;
   (c) $\mathbf{v} = (0, 1, 0)$;
   (d) $\mathbf{v} = (1, 1, 1)$;
   (e) $\mathbf{v} = (-2, 0, 4)$.
2. For (b), (c), and (e) above, can you make sense of the answer you got geometrically?
3. Under what circumstances is $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \mathbf{u}$?
4. Under what circumstances is $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \text{proj}_{\mathbf{u}}(\mathbf{v})$?
5. Finish the proof of Prop 3 above:
   How do we know that $\|\mathbf{u}\| = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$.
6. A particle is traveling along the line $y = 2x$ in $\mathbb{R}^2$ such that its position at time t is $\mathbf{p}(t) = (t, 2t)$.
   (a) At time t, what is the point on the x-axis that the particle is closest to? How far away is the particle from that point?
   (b) At time t, what is the point on the line $y = 5x$ that the particle is closest to? How far away is the particle from that point?

   [Hint: project $\mathbf{p}(t)$ onto the line by taking *any* vector $\mathbf{v}$ (besides $\mathbf{0}$) on the line, and computing $\text{proj}_{\mathbf{v}}(\mathbf{p}(t))$.]