Lecture 20:

More "eigen" stuff
Jordan canonical form

Reminder to finish up the exercises from last time:
(See lecture 19 notes for hints/context.)

1. Last time, you found the eigenvalues of
$$X = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}.$$
Now compute the eigenspaces of each matrix.
[You may assume that $F = \mathbb{R}$ or $\mathbb{C}$.]

2. What are the eigenspaces of $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if we're working over $F = \mathbb{C}$? Does your answer change if we're working over $F = \mathbb{R}$?
[Note: Geometrically, in $\mathbb{R}^2$, multiplication by X acts by rotating clockwise by $\pi/2$. Can you reconcile your answer with this geometric interpretation?]

3. Let $\lambda \in F$. Compute the eigenspaces of
$$X = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and of} \quad Y = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Last time:

Let V be a f.d. vector space over a field F, and let Let $f : V \to V$ be an endomorphism (linear). An **eigenvector** for f is a vector $\mathbf{v} \in F$ such

$$f(\mathbf{v}) = \lambda \mathbf{v}, \quad \text{for some } \lambda \in F.$$

If $\mathbf{v} \neq \mathbf{0}$, then we call λ an **eigenvalue** for f. Since

$$f(\mathbf{v}) = \lambda \mathbf{v} \text{ is equivalent to } (f - \lambda\,\mathrm{id})(\mathbf{v}) = \mathbf{0},$$

the set of eigenvectors of eigenvalue λ, called the **λ-eigenspace** of f and denoted $V_\lambda(f)$, is a vector space (it's the nullspace of $f - \lambda\,\mathrm{id}$).

For any eigenvalue λ, we say the **algebraic multiplicity** of λ is the largest positive integer m_λ such that $(x - \lambda)^{m_\lambda}$ is a factor of $p_f(x)$.

The **geometric multiplicity** of λ is $\boxed{d_\lambda = \dim(V_\lambda(f))}$.

Thm. If λ is an eigenvalue of f, then $1 \leq d_\lambda \leq m_\lambda$.

Eigenbases & diagonalization

We say f is **diagonalizable** if there is a basis $\mathcal{B} = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ of V such that

$$\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

In particular, since $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$, this means that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are all eigenvectors of f. If this is the case, then we're super happy because

$$\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(f^k) = (\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(f))^k = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}.$$

To try to look for an eigenbasis of f,

- find the roots of $p_f(x) = \det(f - x\,\mathrm{id})$ (these are the eigenvalues of f);
- for each root λ, compute the nullspace of $f - \lambda\,\mathrm{id}$ (i.e. compute $V_{\lambda(f)}$)—if f is represented as a matrix X, this is done by row reducing $(X - \lambda\, I \mid \mathbf{0})$; then find a basis $\mathcal{B}_\lambda$ of $V_{\lambda(f)}$; and
- let $S = \bigcup_\lambda \mathcal{B}_\lambda$.

Claim. S is linearly independent, and S is a basis of V iff $d_\lambda = m_\lambda$ for all λ.

Example. Let $V=\mathbb{R}^5$ and consider the linear function $f_Y: V\to V$ corresponding to

$$Y=\begin{pmatrix}-2&0&0&0&9\\9&7&0&-9&-9\\0&0&-2&0&0\\0&0&0&-2&0\\0&0&0&0&7\end{pmatrix}.$$

Then

$$p_Y(x)=-392-476x-134x^2+23x^3+8x^4-x^5=-(x+2)^3(x-7)^2.$$

So $\Lambda=\{-2,7\}$, with $m_{-2}=3$ and $m_7=2$.

We have (see below for computations)

$$V_{-2}=\mathbb{R}\underbrace{\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}\right\}}_{\mathcal{B}_{-2}} \text{ so } d_{-2}=3 \quad\text{and}\quad V_7=\mathbb{R}\underbrace{\left\{\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}}_{\mathcal{B}_7} \text{ and } d_7=2.$$

Then

$$S=\mathcal{B}_{-2}\cup\mathcal{B}_7=\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}, \quad \text{which is a basis!}$$

V_{-2}: Row reduce

$$\left(\begin{array}{ccccc|c}-2+2&0&0&0&9&0\\9&7+2&0&-9&-9&0\\0&0&-2+2&0&0&0\\0&0&0&-2+2&0&0\\0&0&0&0&7+2&0\end{array}\right)\xmapsto{\cdots}\left(\begin{array}{ccccc|c}1&1&0&-1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)$$

So

$$V_{-2}=\mathcal{N}(Y+2\mathrm{id})=\left\{\begin{pmatrix}-x_2+x_4\\x_2\\x_3\\x_4\\0\end{pmatrix}\ \middle|\ x_2,x_3,x_4\in\mathbb{R}\right\}=\mathbb{R}\underbrace{\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}\right\}}_{\mathcal{B}_{-2}}.$$

V_7: Row reduce

$$\left(\begin{array}{ccccc|c}-2-7&0&0&0&9&0\\9&7-7&0&-9&-9&0\\0&0&-2-7&0&0&0\\0&0&0&-2-7&0&0\\0&0&0&0&7-7&0\end{array}\right)\xmapsto{\cdots}\left(\begin{array}{ccccc|c}1&0&0&0&-1&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)$$

So

$$V_7=\mathcal{N}(Y-7\,\mathrm{id})=\left\{\begin{pmatrix}x_5\\x_2\\0\\0\\x_5\end{pmatrix}\ \middle|\ x_2,x_5\in\mathbb{R}\right\}=\mathbb{R}\underbrace{\left\{\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}}_{\mathcal{B}_7}.$$

Example. Let $V = \mathbb{R}^5$ and consider the linear function $f_Y : V \to V$ corresponding to

$$Y = \begin{pmatrix} -1 & 1 & 0 & -1 & 9 \\ 9 & 6 & 0 & -8 & -10 \\ 1 & 0 & -2 & 0 & -1 \\ 1 & 0 & 0 & -2 & -1 \\ 1 & 1 & 0 & -1 & 7 \end{pmatrix}.$$

Then

$$p_Y(x) = -392 - 476x - 134x^2 + 23x^3 + 8x^4 - x^5 = -(x+2)^3(x-7)^2.$$

So $\Lambda = \{-2, 7\}$, with $m_{-2} = 3$ and $m_7 = 2$.

We have (see below for computations)

$$V_{-2} = \mathbb{R}\underbrace{\left\{\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\right\}}_{\mathcal{B}_{-2}} \text{ so } d_{-2} = 2 \quad \text{and} \quad V_7 = \mathbb{R}\underbrace{\left\{\begin{pmatrix}1\\-1\\0\\0\\1\end{pmatrix}\right\}}_{\mathcal{B}_7} \text{ and } d_7 = 1.$$

Then

$$S = \mathcal{B}_{-2} \cup \mathcal{B}_7 = \left\{\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\\1\end{pmatrix}\right\}, \quad \text{which is lin. indep. but not spanning.}$$

V_{-2}: Row reduce

$$\left(\begin{array}{ccccc|c} -1+2 & 1 & 0 & -1 & 9 & 0 \\ 9 & 6+2 & 0 & -8 & -10 & 0 \\ 1 & 0 & -2+2 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2+2 & -1 & 0 \\ 1 & 1 & 0 & -1 & 7+2 & 0 \end{array}\right) \xmapsto{\cdots} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So

$$V_{-2} = \mathcal{N}(Y + 2\,\mathrm{id}) = \left\{\begin{pmatrix}0\\x_4\\x_3\\x_4\\0\end{pmatrix} \middle|\ x_3, x_4 \in \mathbb{R}\right\} = \mathbb{R}\underbrace{\left\{\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\right\}}_{\mathcal{B}_{-2}}. \quad \boxed{d_{-2} = 2}$$

V_7: Row reduce

$$\left(\begin{array}{ccccc|c} -1-7 & 1 & 0 & -1 & 9 & 0 \\ 9 & 6-7 & 0 & -8 & -10 & 0 \\ 1 & 0 & -2-7 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2-7 & -1 & 0 \\ 1 & 1 & 0 & -1 & 7-7 & 0 \end{array}\right) \xmapsto{\cdots} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So

$$V_7 = \mathcal{N}(Y - 7\,\mathrm{id}) = \left\{\begin{pmatrix}x_5\\-x_5\\0\\0\\x_5\end{pmatrix} \middle|\ x_5 \in \mathbb{R}\right\} = \mathbb{R}\underbrace{\left\{\begin{pmatrix}1\\-1\\0\\0\\1\end{pmatrix}\right\}}_{\mathcal{B}_7}. \quad \boxed{d_7 = 1}$$

Do eigenvalues even exist?

We saw last time that some linear transformations don't have eigenvalues.

Fundamental theorem of algebra.
Every polynomial in $\mathbb{C}[x]$ completely factors with roots in $\mathbb{C}$; i.e.

$$\text{if } p(x) \in \mathbb{C}[x] \quad \text{then} \quad p(x) = c(x - \lambda_1) \cdots (x - \lambda_n),$$

for some (not necessarily distinct) $c, \lambda_i \in \mathbb{C}$.

Cor. If $f : V \to V$, where V is a vector space over $F = \mathbb{C}$, then (counting multiplicity) f has $\dim(V)$ eigenvalues. Meaning, if Λ is the set of eigenvalues of f, then

$$\sum_{\lambda \in \Lambda} m_\lambda = \dim(V).$$

Thm. If $p(x) \in \mathbb{R}[x]$, then $p(x)$ factors into polynomials in $\mathbb{R}[x]$ of degree at most 2; i.e.

$$p(x) = c \left(\prod_j (x^2 + a_j x + b_j) \right) \left(\prod_i (x - \lambda_i) \right)$$

for some (not necessarily distinct) $c, \lambda_i, a_j, b_j \in \mathbb{C}$.

Cor. If $f : V \to V$, where V is a vector space over $F = \mathbb{R}$ and $\dim(V)$ is odd, then f has at least one eigenvalue.

If V is a v.s. $/\mathbb{C}$, then $f : V \to V$ has $\dim(V)$ eigenvalues (counting multiplicity. If V is a v.s. $/\mathbb{R}$, and $\dim(V)$ is odd, then $f : V \to V$ has at least one eigenvalue. Otherwise, f may not have any eigenvalues.

A field F is called **algebraically closed** if every polynomial $p \in F[x]$ completely factors with roots in F. Only familiar example: $\mathbb{C}$.

By definition, if λ is an eigenvalue of f, then $V_\lambda(f)$ is non-trivial (there's at least one non-zero eigenvector or eigenvalue λ).

Can eigenspaces overlap (nontrivially)? (No!)

Suppose $\mathbf{v} \in V_\lambda(f)$ and $\mathbf{v} \in V_\mu(f)$. Then

$$\lambda \mathbf{v} = f(\mathbf{v}) = \mu \mathbf{v}.$$

By Midterm 1, this implies that either $\mathbf{v} = \mathbf{0}$ or $\lambda = \mu$.

Lemma. If $\lambda \neq \mu$, then $V_\lambda(f) \cap V_\mu(f) = 0$.

Prop. Let Λ be the set of eigenvalues of f. If, for each $\lambda \in \Lambda$, $\mathcal{B}_\lambda$ is a basis of $V_\lambda(f)$, then

$$S = \bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda \quad \text{is linearly independent.}$$

Pf. Show by induction on ℓ that $\mathbf{v} = c_1 \mathbf{s}_1 + \cdots + c_\ell \mathbf{s}_\ell$ is an eigenvector if and only if all of the $\mathbf{s}_i$ are from the same $\mathcal{B}_\lambda$.

Generalized eigenspaces and Jordan form

What happens if $f : V \to V$ *isn't* diagonalizable?

Let $\lambda \in \Lambda$. Recall that the eigenspace associated to λ is

$$V_\lambda(f) = \{\mathbf{v} \in V \mid (f - \lambda \,\mathrm{id})(\mathbf{v}) = 0\}.$$

The **generalized eigenspace** of eigenvalue λ is

$$V^\lambda(f) = \{\mathbf{v} \in V \mid (f - \lambda \,\mathrm{id})^\ell(\mathbf{v}) = 0 \text{ for some } \ell \in \mathbb{Z}_{\geq 0}\}.$$

Example. Let

$$Y = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad \text{Then } p_Y(x) = -(x-3)^2(x-2),$$

$$V_3(Y) = F\{\mathbf{e}_1\} \subseteq V^3(f) \quad \text{and} \quad V_2(f) = F\{\mathbf{e}_3\} \subseteq V^2(f).$$

But now, note that

$$(Y - 3I_3)\mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so that

$$(Y - 3I_3)^2\mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \text{Hence } \mathbf{e}_2 \in V^3(f) \text{ as well.}$$

What about linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$?

$$(Y - 3I_3)(a\mathbf{e}_1 + b\mathbf{e}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix},$$

so that

$$(Y - 3I_3)^2(a\mathbf{e}_1 + b\mathbf{e}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $F\{\mathbf{e}_1, \mathbf{e}_2\} \subseteq V^3(f)$.

Let $f : V \to V$, where V is a f.d. vector space over $\mathbb{C}$. Let Λ be the set of eigenvalues of f. Let

$$V_\lambda(f) = \{\mathbf{v} \in V \mid (f - \lambda \text{ id})(\mathbf{v}) = 0\}.$$

and

$$V^\lambda(f) = \{\mathbf{v} \in V \mid (f - \lambda \text{ id})^\ell(\mathbf{v}) = 0 \text{ for some } \lambda \in \mathbb{Z}_{\geq 0}\}.$$

Note $V_\lambda(f) \subseteq V^\lambda(f)$.

Homework.

1. $V^\lambda(f)$ is a subspace of V.
2. If $\lambda \neq \mu$, then $V^\lambda(f) \cap V^\mu(f) = 0$.

Claim.

1. For all $\mathbf{v} \in V^\lambda(f)$, $f(\mathbf{v}) \in V^\lambda(f)$.
2. $\dim(V^\lambda(f)) = m_\lambda$.
3. If, for each $\lambda \in \Lambda$, $\mathcal{B}_\lambda$ is a basis of $V^\lambda(f)$, then $\bigcup_{\lambda \in \Lambda}$ is a basis of V.

Theorem. For each $\lambda \in \Lambda$, there is a basis $\mathcal{B}_\lambda$ of $V^\lambda(f)$ for which the matrix representation Y of f (restricted to $V^\lambda(f)$) with respect to $\mathcal{B}_\lambda$ satisfies...

- Y is upper-triangular,
- Y has λ's on the main diagonal,
- some of the entries just above the main diagonal are 1's, and
- all other entries are 0's.

Jordan canonical form

For $\lambda \in F$, we call a $k \times k$ matrix of the form

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ & & \ddots & 1 & & \\ & 0 & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

a $k \times k$ elementary Jordan matrix (of eigenvalue λ). A matrix J is said to be in Jordan canonical form if it consists of Jordan blocks along the diagonal and 0's elsewhere, i.e.

$$J = \operatorname{diag}(J_{k_1}(\lambda_1), \ldots, J_{k_\ell}(\lambda_\ell)) = \begin{pmatrix} \boxed{J_{k_1}(\lambda_1)} & & 0 \\ & \ddots & \\ 0 & & \boxed{J_{k_\ell}(\lambda_\ell)} \end{pmatrix}.$$

Example: $\operatorname{diag}(J_3(7), J_2(7), J_2(-2), J_2(-2), J_1(-2), J_1(-2))$

$$= \begin{pmatrix} 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Thm. Let $Y \in M_n(\mathbb{C})$. Then there is some matrix J in Jordan canonical form such that $J \sim Y$; i.e. there is some choice of basis under which Y can be written in Jordan canonical form. Moreover, this form is unique up to permutation of the blocks.

Pf. Choose "nice" bases of $V^\lambda(Y)$ and put them together.

Some notes:

- The reading comes at Jordan form from a different perspective: polynomials satisfied by the matrix (!!!)—this is an *awesome* topic, and I highly recommend it.

  Example. We saw $Y = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has characteristic polynomial
  $$p_Y(x) = -(x-3)^2(x-2).$$
  But now, notice
$$\begin{aligned} p_Y(Y) &= -(Y - 3\,\mathrm{id})^2(Y - 2\,\mathrm{id}) \\ &= -\begin{pmatrix} 3-3 & 1 & 0 \\ 0 & 3-3 & 0 \\ 0 & 0 & 2-3 \end{pmatrix}^2 \begin{pmatrix} 3-2 & 1 & 0 \\ 0 & 3-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$
- The Jordan blocks in the book are the transpose of these Jordan blocks.
  - Does it matter?
    Not really: They just vary by reversing the order of your favorite basis.
  - Then did we do it *this* way then?
    Our favorite convention has been upper-triangular (rather than lower-triangular).

Lecture 19 end exercises:

1.

$$X = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \qquad \text{we found } p_X(x) = (x-4)^2.$$

Eigenspace

$\lambda = 4$: $\quad V_4 = N(X - 4I_2) = N\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}.$

Reduce
$$\left(\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right) \xrightarrow[r_2 \mapsto \frac{1}{4} r_2]{r_1 \mapsto \frac{1}{6} r_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 1 & -3/2 & 0 \end{array}\right)$$
$$\xrightarrow{r_2 \mapsto r_2 - r_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $\quad V_4 = \left\{ \begin{pmatrix} \left(\frac{3}{2}\right) x_2 \\ x_2 \end{pmatrix} \,\middle|\, x_2 \in F \right\} = F\left\{ \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \right\}$

$$Z = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix} \qquad \text{we found } p_Z(x) = -x(x-2)^2$$

Eigenspaces

$\lambda = 0$: $V_0 = N(Z - 0I) = N(Z).$

Row reduce
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right) \xrightarrow[r_3 \mapsto r_3 + r_1]{r_2 \mapsto r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right) \xrightarrow[r_3 \mapsto -\frac{1}{4} r_3]{r_2 \mapsto \frac{1}{4} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$
$$\xrightarrow[r_3 \mapsto r_3 - r_1]{r_1 \mapsto r_1 - 2r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad \text{So } V_0 = \left\{ \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} \,\middle|\, x_3 \in F \right\} = F\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$\lambda = 2$: $V_2 = N(Z - 2I) = N\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix}.$

Row reduce
$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 2 & -2 & -2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[r_3 \mapsto r_3 - r_1]{r_2 \mapsto r_2 + 2r_1} \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 \mapsto r_1 - r_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow[r_2 \mapsto \frac{1}{2} r_2]{r_1 \mapsto -r_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad \text{so } V_2 = \left\{ \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} \,\middle|\, x_3 \in F \right\} = F\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: then $p_X(x) = \det\begin{pmatrix} -x & 1 \\ -1 & -x \end{pmatrix} = x^2 - (-1) = \underline{x^2+1}$.

If $F = \mathbb{C}$, $p_X(x) = (x-i)(x+i)$ so $\Lambda = \{i, -i\}$ are the eigenvalues.

Eigenspaces

$\lambda = i$: $\left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \xrightarrow[r_2 \mapsto -r_2]{r_1 \mapsto i r_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{r_2 \mapsto r_2 - r_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$V_i = \left\{ \begin{pmatrix} -ix_2 \\ x_2 \end{pmatrix} \middle| x_2 \in \mathbb{C} \right\} = \mathbb{C}\left\{\underline{\begin{pmatrix} -i \\ 1 \end{pmatrix}}\right\}$$

$\lambda = -i$: $\left(\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right) \xrightarrow[r_2 \mapsto i r_2]{r_1 \mapsto -i r_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{r_2 \mapsto r_2 - r_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$V_{-i} = \left\{ \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} \middle| x_2 \in \mathbb{C} \right\} = \mathbb{C}\left\{\underline{\begin{pmatrix} i \\ 1 \end{pmatrix}}\right\}.$$

If $F = \mathbb{R}$, then $p_X(x)$ does not factor. $\Rightarrow$ no e.vals. no e.vects.

3. $X = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$: $p_X(x) = (x-\lambda)^2$ and $V_\lambda = F\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

$Y = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$: $p_Y(x) = -(x-\lambda)^3$ and $V_\lambda = F\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.