Lecture 19:

Eigenvalues and eigenvectors
Diagonalizability
Characteristic polynomial
Eigenspaces

Warmup

1. Compute the determinants of

$$X = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}, \quad Y = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}.$$

2. Let $\lambda \in F$. Compute the determinants of

$$X = \begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix}, \quad Y = \begin{pmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{pmatrix},$$

$$\text{and} \quad Z = \begin{pmatrix} 1-\lambda & 2 & 1 \\ 2 & -\lambda & -2 \\ -1 & 2 & 3-\lambda \end{pmatrix}.$$

[Your answers should be in terms of λ. Reality check: evaluate your answers here $\lambda = 0$, and compare to your answers to 1.]

Answers to warmup:

1. $\det\begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix} = 10(-2)-(-9)4 = 16;$

   $\det\begin{pmatrix}-2 & -1\\ 5 & 2\end{pmatrix} = (-2)(2)-(-1)(5) = 1;$ and

   $\det\begin{pmatrix}1 & 2 & 1\\ 2 & 0 & -2\\ -1 & 2 & 3\end{pmatrix} = (\text{-}1)^{2+1}(2)\det\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix} + (\text{-}1)^{2+2}(0)\det\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$

   $+ (\text{-}1)^{2+3}(-2)\det\begin{pmatrix}1 & 2\\ -1 & 2\end{pmatrix}$ (expanding along row 2)

   $= -2((2)(3)-(1)(2)) + 0 - (-2)((1)(2)-(2)(-1)) = 0.$

2. $\det\begin{pmatrix}10-\lambda & -9\\ 4 & -2-\lambda\end{pmatrix} = (10-\lambda)(-2-\lambda)-(-9)4 = \lambda^2-8\lambda+16;$

   $\det\begin{pmatrix}-2-\lambda & -1\\ 5 & 2-\lambda\end{pmatrix} = (-2-\lambda)(2-\lambda)-(-1)(5) = \lambda^2+1;$ and

$\det\begin{pmatrix}1-\lambda & 2 & 1\\ 2 & -\lambda & -2\\ -1 & 2 & 3-\lambda\end{pmatrix}$ (expanding along row 1)

$= (\text{-}1)^{1+1}(1\text{-}\lambda)\det\begin{pmatrix}\text{-}\lambda & -2\\ 2 & 3\text{ - }\lambda\end{pmatrix} + (\text{-}1)^{1+2}(2)\det\begin{pmatrix}2 & \text{-}2\\ \text{-}1 & 3\text{ - }\lambda\end{pmatrix} + (\text{-}1)^{1+3}(1)\det\begin{pmatrix}2 & \text{-}\lambda\\ \text{-}1 & 2\end{pmatrix}$

$= (1\text{-}\lambda)(\text{-}\lambda(3\text{-}\lambda) \text{ - } (\text{-}2)(2)) - 2(2(3\text{-}\lambda) \text{ - } (\text{-}1)(\text{-}2)) + (2(2) \text{ - } (\text{-}\lambda)(\text{-}1))$

$= -\lambda^3 + 4\lambda^2 - 4\lambda.$

Eigenvectors and eigenvalues

Let V be a f.d. vector space over a field F, and let $f : V \to V$ be an endomorphism (linear). An **eigenvector** for f is a vector $\mathbf{v} \in V$ such that f only scales $\mathbf{v}$ (the direction doesn't change):

$$f(\mathbf{v}) = \lambda\mathbf{v}, \quad \text{for some } \lambda \in F.$$

If $\mathbf{v} \neq \mathbf{0}$, then we call λ an **eigenvalue** for f.

[Root: *eigen* is a German word meaning "belonging to" or "inherent to".]

Similarly, for any matrix $X \in M_n(F)$, eigenvectors and eigenvalues of X are the same as eigenvectors and eigenvalues of the associated endomorphism on F^n (with respect to the standard ordered basis).

Example. For any $X \in M_n(F)$ and any $\lambda \in F$, we have

$$X\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}.$$

So $\mathbf{0}$ is an eigenvector of any matrix. But this is exactly why we require that there's some *nonzero* $\mathbf{v}$ satisfying $X\mathbf{v} = \lambda\mathbf{v}$ to call λ an eigenvalue of X.

(We want being an eigenvalue to be special.)

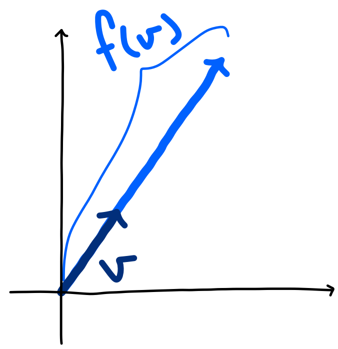
Eigenvectors and eigenvalues

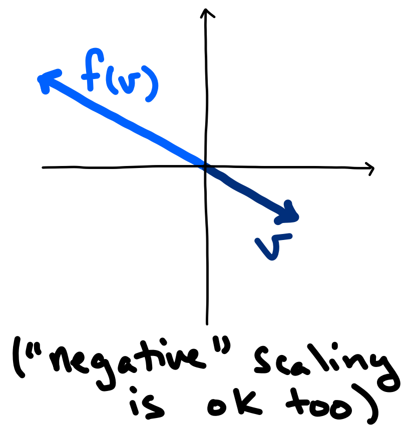
Let V be a f.d. vector space over a field F, and let Let $f : V \to V$ be an endomorphism (linear). An **eigenvector** for f is a vector $\mathbf{v} \in F$ such that f only scales $\mathbf{v}$ (the direction doesn't change):

$$f(\mathbf{v}) = \lambda \mathbf{v}, \quad \text{for some } \lambda \in F.$$

If $\mathbf{v} \neq \mathbf{0}$, then we call λ an **eigenvalue** for f.

[Root: *eigen* is a German word meaning “belonging to” or “inherent to”.]





Eigenvectors and eigenvalues

Let V be a f.d. vector space over a field F, and let Let $f : V \to V$ be an endomorphism (linear). An **eigenvector** for f is a vector $\mathbf{v} \in F$ such that f only scales $\mathbf{v}$ (the direction doesn't change):

$$f(\mathbf{v}) = \lambda \mathbf{v}, \quad \text{for some } \lambda \in F.$$

If $\mathbf{v} \neq \mathbf{0}$, then we call λ an **eigenvalue** for f.

[Root: *eigen* is a German word meaning “belonging to” or “inherent to”.]

Similarly, for any matrix $X \in M_n(F)$, eigenvectors and eigenvalues of X are the same as eigenvectors and eigenvalues of the associated endomorphism on F^n (with respect to the standard ordered basis).

Example. For any $X \in M_n(F)$ and any $\lambda \in F$, we have

$$X\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}.$$

So $\mathbf{0}$ is an eigenvector of any matrix. But this is exactly why we require that there's some *nonzero* $\mathbf{v}$ satisfying $X\mathbf{v} = \lambda\mathbf{v}$ to call λ an eigenvalue of X.

(We want being an eigenvalue to be special.)

Example. Let

$$X = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then

$$X\mathbf{u} = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \mathbf{2}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \mathbf{2u};$$

and

$$X\mathbf{v} = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \mathbf{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{3v}.$$

So $\mathbf{2}$ and $\mathbf{3}$ are eigenvalues of X, and $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of X (of eigenvalue 2 and 3, respectively).

Notice: $\mathcal{B} = \left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$ is a basis of F^2. And with respect to this basis, the above calculations show

$$\mathrm{Rep}_{\mathcal{B}}^{\mathcal{B}}(f_X) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

where $f_X : F^2 \to F^2$ is the linear function associated to X.
In particular,

$$X = PDP^{-1} \quad \text{where } D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

Why is this great? For example, what if we wanted to compute X^{100}?

Theorem. Let $X \in M_n(F)$ with corresponding linear function $f_X : F^n \to F^n$. Suppose $\mathcal{B} = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ is an ordered basis of eigenvectors with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e.,

$$X\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \text{for } i = 1, \ldots, n.$$

Let P be the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and let

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then

$$X = PDP^{-1}.$$

In this setting, we say X is **diagonalizable**.

One reason to care: Usually, matrix multiplication is computationally expensive, unless the matrices are very **sparse** (have lots of 0's). But diagonal matrices are *very* sparse! So if X is diagonalizable, then

$$X^\ell = (PDP^{-1})^\ell = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^\ell P^{-1};$$

and $D^\ell = \mathrm{diag}(\lambda_1^\ell, \ldots, \lambda_n^\ell)$. (By calculation!)

Important questions:

- When does an endomorphism even have such a nice basis?
- How do we find eigenvectors and eigenvalues?

Finding eigenvalues and eigenvectors.

For a matrix $X \in M_n(F)$ and a vector $\mathbf{v} \in F^n$, note that

$$X\mathbf{v} = \lambda\mathbf{v} \quad \text{if and only if} \quad \mathbf{0} = X\mathbf{v} - \lambda\mathbf{v} = (X - \lambda I_n)\mathbf{v}.$$

So

$$X\mathbf{v} = \lambda\mathbf{v} \quad \text{if and only if} \quad \mathbf{v} \in \mathcal{N}(X - \lambda I_n).$$

Again, $\mathbf{v} = \mathbf{0}$ is always a solution. But we're interested in *non-trivial* solutions! So $\lambda \in F$ is an eigenvalue for X if and only if $\mathcal{N}(X - \lambda I_n) \neq \{0\}$.

Determinant to the rescue!!!

$$\ker(X - \lambda I_n) \neq \{0\} \quad \Leftrightarrow \quad \operatorname{rank}(X - \lambda I_n) < n \quad \Leftrightarrow \quad \det(X - \lambda I_n) = 0.$$

To find the eigenvalues of X, solve $\det(X - \lambda I_n) = 0$ for $\lambda \in F$.

Example. Back to $X = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$: We have

$$X - \lambda I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix}.$$

So

$$\det(X - \lambda I_2) = (-1-\lambda)(6-\lambda) - 2(-6) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Thus $\det(X - \lambda I_2) = 0$ when $\lambda = 2$ or $\lambda = 3$;

and hence these are exactly the two eigenvalues of X.

You try:

1. Find the eigenvalues of

$$X = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}, \quad Y = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}.$$

   (See warmup.) Does it matter what F is?

2. If

$$X = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \quad \text{then } \det(X - \lambda I_4) = \lambda(\lambda - 2)(\lambda - 4)^2.$$

   This means that X has eigenvalues $\lambda = 0$, $\lambda = 2$, and $\lambda = 4$.

   Compute the nullspaces of $X - 0I_4$ (this matrix is just X) and of $X - 2I_4$. (The computation of the nullspace of $X - 4I_4$ is on the next page.)

First, we have

$$X - 4I_4 = \begin{pmatrix} 2-4 & -1 & -1 & 0 \\ -1 & 3-4 & -1 & -1 \\ -1 & -1 & 3-4 & -1 \\ 0 & -1 & -1 & 2-4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -2 \end{pmatrix}.$$

Recall that to compute the nullspace of $X - 4I_4$, we should row reduce:

$$\left(\begin{array}{cccc|c} -2 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -2 & 0 \end{array}\right) \xrightarrow[\text{row}_3 \mapsto \text{row}_3 - \text{row}_2]{\text{row}_1 \mapsto \text{row}_1 - 2\text{row}_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[\text{row}_4 \mapsto \text{row}_4 + \text{row}_1]{\text{row}_2 \mapsto -\text{row}_2} \left(\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{row}_1 \leftrightarrow \text{row}_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{row}_1 \mapsto \text{row}_1 - \text{row}_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \quad \text{So } \begin{cases} x_1 - x_4 = 0, \\ x_2 + x_3 + 2x_4 = 0. \end{cases}$$

Thus, $x_1 = x_4$ and $x_2 = -x_3 - 2x_4$, so that

$$\mathcal{N}(X - 4I_2) = \left\{ \begin{pmatrix} x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} \middle| \, x_3, x_4 \in F \right\} = F\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Continuing with the example from Problem 2:

We just saw that matrix $X = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ has eigenvalues $\lambda = 0,\ 2,$ and 4; and that

$$\mathcal{N}(X - 0I_4) = F\{(1,1,1,1)^t\},$$
$$\mathcal{N}(X - 2I_4) = F\{(-1,0,0,1)^t\}, \quad \text{and}$$
$$\mathcal{N}(X - 4I_4) = F\{(0,-1,1,0)^t, (1,-2,0,1)\}.$$

One can check that

$$\mathcal{B} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

is a basis of F^4 (so long as F isn't too small). So

$$X = PDP^{-1} \quad \text{where } P = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{4} & 0 \\ 0 & 0 & 0 & \mathbf{4} \end{pmatrix}.$$

General strategy:

- Find the eigenvalues of X by solving $\det(X - \lambda I_n) = 0$ for λ.
- For each eigenvalue λ, compute a basis for $\mathcal{N}(X - \lambda I_n)$.
- If this process results in finding n eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then A is **diagonalizable**. (This is a *significant* "if", but there's a reasonable backup plan.)

Let V be a finite-dimensional vector space over F, and let $f : V \to V$ be an endomorphism. Back to the language of linear maps (instead of just matrices):

- The eigenvalues of f are the roots of $\det(f - \lambda \text{ id})$. Note that the determinant $\det(f - \lambda \text{ id})$ will always be a polynomial in the variable λ. We call
$$p_f(x) = \det(f - x \text{ id})$$
the **characteristic polynomial** of f.
(The eigenvalues of f are exactly the roots of $p_f(x)$.)
- The eigenvectors of f associated to λ are those vectors in $\mathcal{N}(f - \lambda \text{ id})$. We call this space the **eigenspace** of f corresponding to λ, or just the λ-eigenspace of f, denoted
$$V_\lambda = V_\lambda(f) = \{\mathbf{v} \in V \mid f(\mathbf{v}) = \lambda \mathbf{v}\}.$$
Note that because V_λ is a nullspace, it is a subspace of V.
- For $\lambda \in F$, if $(x - \lambda)^\ell$ is a factor of $p_f(x)$, but $(x - \lambda)^{\ell+1}$ is not, we call ℓ the **(algebraic) multiplicity** of λ, and denote it $\ell = m_\lambda$.

Thm. If λ is an eigenvalue of f, then $1 \leq \dim(V_\lambda(f)) \leq m_\lambda$.

- If there is a basis of V consisting of eigenvectors of f, we call such a basis an **eigenbasis**, and say that f is **diagonalizable** (since there is a basis in which f is represented as a diagonal matrix).
Sufficient (but not necessary): $\dim(V_\lambda(f)) = 1$ for all eigenvalues λ.

You try:

1. Above, you found the eigenvalues of
$$X = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}.$$
Now compute the eigenspaces of each matrix.
[For X, the eigenspaces are subspaces of F^2; the eigenspaces of Z are subspaces of F^3. You may assume that $F = \mathbb{R}$ or $\mathbb{C}$.]

2. What are the eigenspaces of $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if we're working over $F = \mathbb{C}$?
[Compute $p_X(x)$ and find its roots; then compute $V_\lambda(X) = \mathcal{N}(X - \lambda I_2)$ for each root λ.]

   Does your answer change if we're working over $F = \mathbb{R}$?
[Note: Geometrically, in $\mathbb{R}^2$, multiplication by X acts by rotating clockwise by $\pi/2$. Can you reconcile your answer with this geometric interpretation?]

3. Let $\lambda \in F$. Compute the eigenspaces of
$$X = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and of} \quad Y = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$