## Lecture 18:

More determinants!
Permutation expansion
Laplace's cofactor expansion

## Big takeaways from the worksheet:

If a determinant $\det : M_n(F) \to F$ exists...

- $\det(X) = 0$ if and only if $X$ is singular (non-invertible);
- $\det$ is multiplicative (meaning that $\det(XY) = \det(X)\det(Y)$);
- $\det(X^t) = \det(X)$.

(You may take these as theorems now, whose proofs are outlined in the Lecture 17 worksheet.)

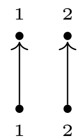
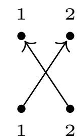
The main obstruction to a determinant existing is if two sequences of row operations accidentally give us different results.

**Goal:** Find a closed formula for determinant using multilinearity, and then show *that* formula defines a determinant. (Check that the formula itself is normalized, alternating, and multilinear.)

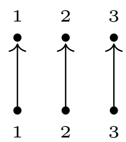
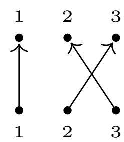
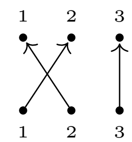
A **permutation** is just another word for a bijective function (usually on a finite set), but thought of a little differently. Let $[n]$ denote the set $\{1, 2, \dots\}$. Let

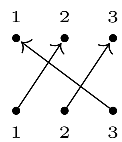
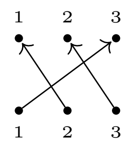
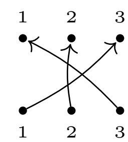
$$S_n = \left\{ \sigma : [n] \to [n] \;\middle|\; \sigma \text{ is bijective } \right\}$$

be the set of **permutations of** $[n]$. For example, the permutations of $[2]$ are

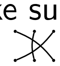
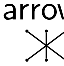
$\mathrm{id} =$  meaning $\mathrm{id}(1) = 1$, $\mathrm{id}(2) = 2$; and $\sigma =$  meaning $\sigma(1) = 2$, $\sigma(2) = 1$.

The permutations of $[3]$ are

, , ,

, , and .

We call this pictures of permutations **permutation diagrams**.

[Note: I went to a little trouble to make sure all the arrows didn't cross at the same point in that last permutation:  versus .
This is a useful precaution in times to come.]

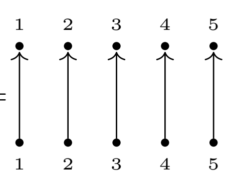
## Sign of a permutation

The **sign** of a permutation $\sigma \in S_n$ is

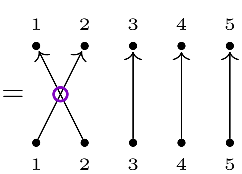
$$\mathrm{sgn}(\sigma) = (-1)^{\#\{\text{crossings}\}} = (-1)^{\#\{\text{inversions}\}}$$

where $\sigma$ is drawn with at most two edges crossing at any point.

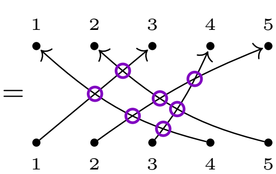
For example, if $n = 5$:

$\mathrm{id} =$ 

# intersections: 0
$\mathrm{sgn}(\mathrm{id}) = (-1)^0 = 1$

$\sigma =$ 

# intersections: 1
$\mathrm{sgn}(\sigma) = (-1)^1 = -1$

$\tau =$ 

# intersections: 7
$\mathrm{sgn}(\tau) = (-1)^7 = -1$

The crossings in a diagram of a permutation $\sigma \in S_n$ are really just detecting **inversion pairs**:

$$(i, j) \quad \text{such that} \quad i < j \quad \text{but} \quad \sigma(i) > \sigma(j).$$

For example, in $\tau$ above, the inversion pairs are

$$(1,4), \quad (1,5), \quad (2,3), \quad (2,4), \quad (2,5), \quad (3,4), \quad \text{and} \quad (3,5).$$

[To find inversions $(i, j)$, look for the arrows $j \to \tau(j)$ that cross $i \to \tau(i)$ from SE to NW.]

# Permutation matrices

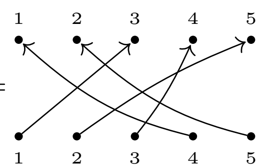
For a permutation $\sigma \in S_n$, we define the **permutation matrix** $P_\sigma$ as the linear extension of the map

$$\mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}; \quad \text{i.e. } P_\sigma \mathbf{e}_i = \mathbf{e}_{\sigma(i)}.$$

Namely, the $i$th column of $P_\sigma$ is $\mathbf{e}_{\sigma(i)}$.

**Example:** If $n = 5$ and

$\tau =$

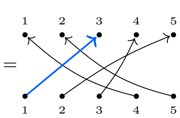



, then $P_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$

**Lemma.** The permutation matrices are those matrices with exactly one 1 in each row and in each column, and 0's elsewhere.

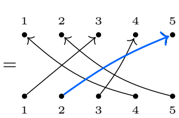
**Proof.** The columns of a permutation matrix are elementary basis vectors; and since a permutation is bijective, each basis vector appears in exactly one column. This observation exactly coincides with the statement of this Lemma.

$\tau =$



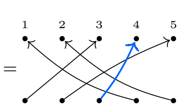
, then $P_\tau = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\tau =$



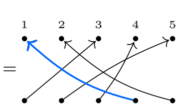
, then $P_\tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\tau =$



, then $P_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\tau =$



, then $P_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

For a permutation $\sigma \in S_n$, we define the **permutation matrix** $P_\sigma$ as the linear extension of the map

$$\mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}; \quad \text{i.e. } P_\sigma \mathbf{e}_i = \mathbf{e}_{\sigma(i)}.$$

Namely, the $i$th column of $P_\sigma$ is $\mathbf{e}_{\sigma(i)}$.

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**Lemma.** The permutation matrices are those matrices with exactly one 1 in each row and in each column, and 0's elsewhere.

**Ex.** In $M_2(F)$, the permutation matrices are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P_{1,2}$.

**Ex.** In $M_3(F)$, the permutation matrices are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P_{1,2}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P_{2,3},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = P_{1,3},$$

(Compare to the six permutations of $[3] = \{1, 2, 3\}$.)

**Lemma.** $\det(P_\sigma) = \operatorname{sgn}(\sigma)$.

**Proof**-ish. To row-reduce $P_\sigma$ to $I_n$, put pivots where they belong from left-to-right, by a sequence of adjacent rows swapst. (First find the row that has $\mathbf{e}_1$ in it, and move it up one row at a time until it's at the top; then find the row that has $\mathbf{e}_2$ and move it up one row at a time until it's at the top, . . . ). Each step "removes" one inversion, and toggles the determinant by a multiple of $-1$.

**Lemma.** For any $\sigma \in S_n$, $\det(P_\sigma) = \operatorname{sgn}(\sigma)$.

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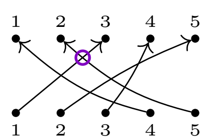
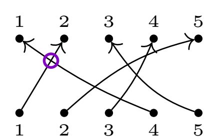
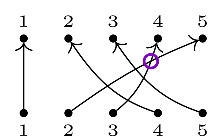
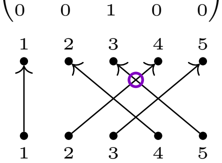
$$P_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \xmapsto{\mathbf{r}_2 \leftrightarrow \mathbf{r}_3} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \xmapsto{\mathbf{r}_1 \leftrightarrow \mathbf{r}_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

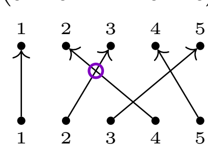


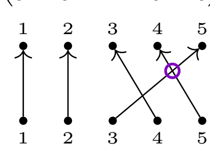



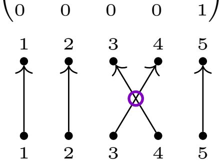

$$\xmapsto{\mathbf{r}_4 \leftrightarrow \mathbf{r}_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \xmapsto{\mathbf{r}_3 \leftrightarrow \mathbf{r}_4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \xmapsto{\mathbf{r}_2 \leftrightarrow \mathbf{r}_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

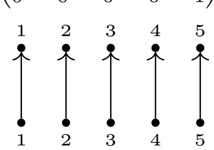



$$\xmapsto{\mathbf{r}_4 \leftrightarrow \mathbf{r}_5} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xmapsto{\mathbf{r}_3 \leftrightarrow \mathbf{r}_4} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (-1)^7 \det(P_\tau) = \det(I_5)$$




So $\det(P_\tau) = (-1)^7 \det(I_5)$.

# Permutation expansion

Use multilinearity to expand determinant!

**Example:** $n = 3$.
Since

$$(x_{1,1}, x_{1,2}, x_{1,3}) = (x_{1,1}, 0, 0) + (0, x_{1,2}, 0) + (0, 0, x_{1,3}),$$

we have

$$\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}.$$

First expand using

$$(x_{1,1}, x_{1,2}, x_{1,3}) = (x_{1,1}, 0, 0) + (0, x_{1,2}, 0) + (0, 0, x_{1,3}).$$

Similarly, expand in row 2 using

$$(x_{2,1}, x_{2,2}, x_{2,3}) = (x_{2,1}, 0, 0) + (0, x_{2,2}, 0) + (0, 0, x_{2,3}):$$

$$\begin{aligned}
\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}
&= \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ 0 & x_{2,2} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ x_{2,1} & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ 0 & x_{2,2} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ 0 & 0 & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}.
\end{aligned}$$

But, for example,

$$\det\begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = 0 \quad \text{because } \mathbf{r}_2 = 0 \text{ or } \mathbf{r}_1 = \frac{x_{1,1}}{x_{2,1}}\mathbf{r}_2.$$

Finally, expand each (non-zero) determinant in the third row, using

$$(x_{3,1}, x_{3,2}, x_{3,3}) = (x_{3,1}, 0, 0) + (0, x_{3,2}, 0) + (0, 0, x_{3,3}):$$

$$\det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \underbrace{\det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ x_{3,1} & 0 & 0 \end{pmatrix}}_{0} + \underbrace{\det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ 0 & x_{3,2} & 0 \end{pmatrix}}_{0} + \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ 0 & 0 & x_{3,3} \end{pmatrix}$$

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$$\det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \underbrace{\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & 0 & 0 \end{pmatrix}}_{0} + \begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ 0 & x_{3,2} & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ 0 & 0 & x_{3,3} \end{pmatrix}}_{0}$$

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Note: In the end, we're only left with terms whose "footprint" is in the shape of a permutation matrix! (Meaning that they're a permutation matrix whose rows have been scaled.)

So

$$\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$
$$= \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ 0 & 0 & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ 0 & x_{3,2} & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \\ 0 & 0 & x_{3,3} \end{pmatrix}$$
$$+ \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ 0 & x_{2,2} & 0 \\ x_{3,1} & 0 & 0 \end{pmatrix}$$
$$= x_{1,1}x_{2,2}x_{3,3}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_{1,1}x_{2,3}x_{3,2}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_{1,2}x_{2,1}x_{3,3}\det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$+ x_{1,2}x_{2,3}x_{3,1}\det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + x_{1,3}x_{2,1}x_{3,2}\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + x_{1,3}x_{2,2}x_{3,1}\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

using the multilinearity of determinant to pull out coefficients one row at a time.

Rewriting this result in terms of permutations, we have

$$\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \sum_{\sigma \in S_3} x_{\sigma(1),1} x_{\sigma(2),2} x_{\sigma(3),3} \underbrace{\det(P_\sigma)}_{\operatorname{sgn}(\sigma)}.$$

**Theorem.** For $X \in M_n$, define

$$\det(X) = \sum_{\sigma \in S_n} x_{\sigma(1),1} x_{\sigma(2),2} \dots x_{\sigma(n),n} \operatorname{sgn}(\sigma)$$

Then $\det : M_n(F) \to F$ is a determinant.

**Proof.** See Ch. Four, §I.4, Lemma 4.9. *Notation:* The book writes $\iota_j$ for $\mathbf{e}_j$.

Caution! We have only justified that if a determinant exists, it must satisfy this formula. To prove that this formula is a determinant (hence showing that determinant is well-defined), you must check that it is normalized, alternating, and multilinear.

**You try:**

1. Use the permutation expansion to compute the determinants of

$$\text{(a)} \quad X = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \text{(b)} \quad Y = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix}.$$

2. Compare the permutation expansion of $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the determinant function we already established for $M_2(F)$.
3. Use the formula in the theorem above to compute $\det(I_n)$ (confirm it's equal to 1 as it should be).

## Reading the inversions directly off of a permutation matrix:

For each 1, count how many 1's are NE of it, and add up those values.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

2 inversions $(1, j)$ 3 inversions $(2, j)$ 2 inversions $(3, j)$

$$\begin{pmatrix} 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

0 inversions $(4, j)$ 0 inversions $(5, j)$

**Total:** $2 + 3 + 2 + 0 + 0 = \boxed{7}$ inversions.

## Answer to 1(b)

$$\det \begin{pmatrix} 1 & 0 & 3 \\ 5 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix}$$

$$= \det \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix} + \det \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & \boxed{4} & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & \boxed{0} & 0 \\ \boxed{5} & 0 & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix}$$

$$+ \det \begin{pmatrix} 0 & 0 & \boxed{3} \\ \boxed{5} & 0 & 0 \\ 0 & \boxed{4} & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & \boxed{0} & 0 \\ 0 & 0 & \boxed{1} \\ \boxed{0} & 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & \boxed{3} \\ 0 & \boxed{2} & 0 \\ \boxed{0} & 0 & 0 \end{pmatrix}$$

$$= (1)(2)(-1)\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1)(4)(1)\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \underbrace{(5)(0)(-1)}_{0} \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ (5)(4)(3)\det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \underbrace{(0)(0)(0)}_{0} \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \underbrace{(0)(2)(3)}_{0} \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= -2(-1)^0 + 4(-1)^1 + 0 + 60 * (-1)^2 + 0 + 0$$

$$= -2 - 4 + 60 = 54.$$

## Grouping terms wisely:

$$\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$

$$\begin{aligned}
&= \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & 0 \\ 0 & 0 & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & 0 & x_{2,3} \\ 0 & x_{3,2} & 0 \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \\ 0 & 0 & x_{3,3} \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & 0 \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & x_{1,2} & 0 \\ 0 & 0 & x_{2,3} \\ x_{3,1} & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & x_{1,3} \\ 0 & x_{2,2} & 0 \\ x_{3,1} & 0 & 0 \end{pmatrix} \\
&= x_{1,1}x_{2,2}x_{3,3} - x_{1,1}x_{3,2}x_{2,3} \\
&\quad - x_{2,1}x_{1,2}x_{3,3} + x_{2,1}x_{3,2}x_{1,3} \\
&\quad + x_{3,1}x_{1,2}x_{2,3} - x_{3,1}x_{2,2}x_{1,3}
\end{aligned}$$

$$\begin{aligned}
&= \det\begin{pmatrix} x_{1,1} & 0 & 0 \\ 0 & x_{2,2} & x_{2,3} \\ 0 & x_{3,2} & x_{3,3} \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & x_{1,2} & x_{1,3} \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \end{pmatrix} \\
&+ \det\begin{pmatrix} 0 & x_{1,2} & x_{1,3} \\ 0 & x_{2,2} & 2,3 \\ x_{3,1} & 0 & 0 \end{pmatrix} \\
&= x_{1,1}(x_{2,2}x_{3,3} - x_{3,2}x_{2,3}) \\
&\quad - x_{2,1}(x_{1,2}x_{3,3} - x_{3,2}x_{1,3}) \\
&\quad + x_{3,1}(x_{1,2}x_{2,3} - x_{2,2}x_{1,3})
\end{aligned}$$

The **$(k,\ell)$-submatrix** of a matrix $X \in M_n(F)$ is the matrix $\mathrm{Sub}_{k,\ell}(X)$ gotten by deleting the $k$th row and $\ell$th column from $X$. For example, taking $X \in M_3(F)$ above, we have

$$\mathrm{Sub}_{1,1}(X) = \begin{pmatrix} x_{2,2} & x_{2,3} \\ x_{3,2} & x_{3,3} \end{pmatrix} \quad \mathrm{Sub}_{2,1}(X) = \begin{pmatrix} x_{1,2} & x_{1,3} \\ x_{3,2} & x_{3,3} \end{pmatrix} \quad \mathrm{Sub}_{3,1}(X) = \begin{pmatrix} x_{1,2} & x_{1,3} \\ x_{2,2} & x_{2,3} \end{pmatrix}.$$

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For example, taking $X \in M_3(F)$ above, we have

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Then

$$\det\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = x_{1,1}\det(\mathrm{Sub}_{1,1}(X)) - x_{2,1}\det(\mathrm{Sub}_{2,1}(X)) + x_{3,1}\det(\mathrm{Sub}_{3,1}(X))$$

We call $(-1)^{k+\ell}\det(\mathrm{Sub}_{k,\ell}(X))$ the **cofactor** of entry $(k,\ell)$.

**Theorem.** (Laplace's cofactor expansion) For any fixed $1 \le k \le n$, we have

$$\det(X) = \sum_{\ell=1}^{n} (-1)^{k+\ell} X_{k,\ell} \det(\mathrm{Sub}_{k,\ell}(X)); \quad \text{(fixed row } k\text{)}$$

and for any fixed $1 \le \ell \le n$, we have

$$\det(X) = \sum_{\ell=1}^{n} (-1)^{k+\ell} X_{k,\ell} \det(\mathrm{Sub}_{k,\ell}(X)). \quad \text{(fixed column } \ell\text{)}$$

Remark: Think of this theorem like a recursive way to reduce determinant calculations. See: Chapter Four, Section III.1 for examples.