LECTURE 17 WORKSHEET: DETERMINANTS

Instructions. Work in groups of 2–4. Appoint a scribe to begin filling in the main packet as you work together; trade off scribes every 20-ish minutes. Hand in whatever you have done at the end of class.

Review

For a function $f : M_n(F) \to F$, we can write f as a multivariable function $f(X) = f(\mathbf{r}_1(X), \ldots, \mathbf{r}_n(X))$, where $\mathbf{r}_1(X), \ldots, \mathbf{r}_n(X)$ are the row vectors of X.

Definition 1. A *determinant* is a function $\det : M_n(F) \to F$ that satisfies the following.

(i) **Normalized.** $\det(I_n) = \det(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

(ii) **Alternating.** Swapping any two rows toggles the sign of the function:

$$\det(\ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots) = -\det(\ldots, \mathbf{r}_j, \ldots, \mathbf{r}_i, \ldots).$$

(iii) **Multilinear.** The determinant is a linear function with respect to every row (individually): for each $i = 1, \ldots, n$, we have

$$\begin{aligned}\det(\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_i + \lambda\, \mathbf{r}_i', \mathbf{r}_{i+1}, \ldots, \mathbf{r}_n) &= \det(\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_i, \mathbf{r}_{i+1}, \ldots, \mathbf{r}_n) \\ &\quad + \lambda \det(\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_i', \mathbf{r}_{i+1}, \ldots, \mathbf{r}_n).\end{aligned}$$

For example, we saw for $n = 2$ that

$$\det : M_2(F) \to F \quad \text{defined by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc \tag{$*$}$$

defines a determinant for 2×2 matrices.

Recall that the elementary row operation matrices in $M_n(F)$ are

$$\text{scaling: } S_i(\lambda) = \lambda E_{i,i} + \sum_{j \neq i} E_{j,j}; \qquad \text{permutation: } P_{i,j} = E_{i,j} + E_{j,i} + \sum_{r \neq i,j} E_{r,r};$$

$$\text{and} \qquad \text{combination: } C_{i,j}(\lambda) = \lambda E_{j,i} + I_n.$$

Lemma 2. *Let $\det : M_n(F) \to F$ be a determinant, and let $X \in M_n$.*

(a) We have

$$\det(S_i(\lambda)X) = \lambda \det(X), \quad \det(P_{i,j}X) = -\det(X), \quad \textit{and} \quad \det(C_{i,j}(\lambda)X) = \det(X).$$

(b) If $\mathbf{r}_i(X) = 0$ for some i, then $\det(X) = 0$.

(c) If $\mathbf{r}_i(X) = \mathbf{r}_j(X)$ for some $i \neq j$, then $\det(X) = 0$.

Theorem 3.

(a) If a determinant $\det : M_n(F) \to F$ exists, then it is unique. (In other words, if $f : M_n(F) \to F$ satisfies Definition 1, we must have $f = \det$.)

(b) If $\det : M_n(F) \to F$ is a determinant, then for any $X \in M_n(F)$, we have $\det(X) = 0$ if and only if X is singular (non-invertible).

You try

1. Warm up. Find two different ways to row reduce $X = \begin{pmatrix} 1 & 3 \\ 5 & -20 \end{pmatrix}$ (two different sequences of steps—aim for something *signifigantly* different). Use both to compute $\det(X)$, and check that you get the same answer. Then check that your answer agrees with Equation $(*)$.

2. Playing with the defining axioms of determinant.

(a) Take a moment to summarize why Theorem 3(a) and (b) both follow from Lemma 2.

(b) Prove that, assuming multilinearity, the condition in Lemma 2(c) is equivalent to the alternating property.

(*A little more.* In class, we showed that the alternating property of determinants implies Lemma 2(c). Conversely, prove that if $f : M_n(F) \to F$ *instead* satisfies multilinearity and the property in Lemma 2(c) (if any two rows are equal then the determinant is 0), then f is alternating.)

3. Determinants are multiplicative. Our goal in this problem is to show that if a determinant $\det : M_n(F) \to F$ exists, then it is multiplicative; i.e.

$$\textit{Claim:} \quad \det(XY) = \det(X)\det(Y) \quad \text{for all} \quad X, Y \in M_n(F).$$

We will do this in two cases: when $\det(Y) \neq 0$ and when $\det(Y) = 0$.

Case 1: $\det(Y) \neq 0$.

Fix some $Y \in M_n(F)$ such that $\det(Y) \neq 0$. Let

$$D_Y : M_n(F) \longrightarrow F \quad \text{defined by} \quad D_Y(X) = \det(XY)/\det(Y).$$

You will prove that $D_Y(X) = \det(X)$ by showing that D_Y is *also* a determinant (by Thm. 3(a)).

Caution! You *do not know what function* det *is*: if I gave you a matrix and asked you to compute det, you don't have a closed formula for that yet. In the following exercises, the *only* things you know about det is what you're told in Definition 1. Just rely on those properties, Lemma 2, and/or Theorem 3 applied to det (but *not* to D_Y—you don't yet know that D_Y is a determinant).

(a) Compute $D_Y(I_n)$.

(b) Prove that if $\mathbf{r}_i(X) = \mathbf{r}_j(X)$, then $\mathbf{r}_i(XY) = \mathbf{r}_j(XY)$, and use this to argue that $D_Y(X) = 0$. Conclude that D_Y is alternating. (For the first part, once you know $\mathbf{r}_i(XY) = \mathbf{r}_j(XY)$, you can use Lemma 2 to say something useful about $\det(XY)$. For the last part, use Problem 2b.)

(c) Prove that D_Y is multilinear on rows. That is, show that D_Y satisfies

$$D_Y(\mathbf{r}_1, \dots, \mathbf{r}_i + \lambda \cdot \mathbf{r}_i', \dots, \mathbf{r}_n) = D_Y(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n) + \lambda D_Y(\mathbf{r}_1, \dots, \mathbf{r}_i', \dots, \mathbf{r}_n)$$

for all $\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{r}_i' \in F^n$ and any $\lambda \in F$.

Hint. Some suggested notation to help in your proof: Let

X be the matrix with rows $\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n$;

X' be the matrix with rows $\mathbf{r}_1, \dots, \mathbf{r}_i', \dots, \mathbf{r}_n$; and

Z be the matrix with rows $\mathbf{r}_1, \dots, \mathbf{r}_i + \lambda \mathbf{r}_i', \dots, \mathbf{r}_n$.

Let $\mathbf{c}_1, \dots, \mathbf{c}_n$ denote the columns of Y. Then $(XY)_{s,t} = \mathbf{r}_s \cdot \mathbf{c}_t$; i.e. the s,t-entry of XY is the dot product of the s-th row of X with the t-th column of Y. You will need compare the rows of XY, $X'Y$ and ZY in order to prove that

$$\frac{\det(XY)}{\det(Y)} + \lambda \frac{\det(X'Y)}{\det(Y)} = \frac{\det(ZY)}{\det(Y)}.$$

(d) Deduce that for all X, we have that $D_Y(X) = \det(X)$. Use this to show that $\det(XY) = \det(X)\det(Y)$. (Use Theorem 3(a) for the first part.)

$\boxed{\textbf{Case 2: } \det(Y) = 0.}$

Let $Y \in M_n(F)$ satisfy $\det(Y) = 0$. Prove that $\det(XY) = \det(X)\det(Y)$.

(*Hint:* We have basically done this already. Explain why $\det(Y) = 0$ if and only if $\operatorname{rank}(Y) < n$ (See Theorem 3(b)); and then consider $\operatorname{rank}(XY)$.)

4. Transpose. Assume a determinant $\det : M_n(F) \to F$ exists.

(a) Recall that matrix inversion reverses the order of multiplication: for any invertible $X, Y \in M_n(F)$, we have $(XY)^{-1} = Y^{-1}X^{-1}$. We aim to prove a similar result for transpose: for $X \in M_{k,\ell}(F)$ and $Y \in M_{\ell,n}(F)$, we have

$$(XY)^t = Y^t X^t.$$

(i) Compute $(XY)^t$ and $Y^t X^t$ for

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix},$$

and compare. (Compute XY and take the transpose; then compute Y^t and X^t and multiply.)

(ii) Prove using direct computation that $(XY)^t = Y^t X^t$ for any $X \in M_{k,\ell}(F)$ and $Y \in M_{\ell,n}(F)$. (Hint: If the rows and columns of X are $\mathbf{r}_i(X)$ and $\mathbf{c}_i(X)$, respectively, what are the rows and columns of X^t? Compute the (i, j)-entry of $X^t Y^t$ and of $(XY)^t$ and compare.)

(b) Use the previous problem to show that for any invertible $X \in M_n(F)$, we have

$$(X^{-1})^t = (X^t)^{-1}.$$

(Multiply $(X^{-1})^t$ by X^t and use the previous problem.)

(c) Check that for each of the elementary row operation matrices $R \in M_n(F)$, we have $\det(R) = \det(R^t)$ by row reducing R^t to the identity.

(d) Show that for $X \in M_n(F)$, we have $\det(X^t) = \det(X)$.
(There is a sequence of elementary row operation matrices $R_1, \ldots, R_\ell$ such that $R_\ell \cdots R_1 X = E$, where E is in reduced row echelon form. Then (1) compute $\det(X)$ in terms of the $\det(R_i)$'s and $\det(E)$ (which is either 0 or 1 depending on whether or not X has full rank); and (2) compute $\det(X^t)$ by taking the transpose of $R_\ell \cdots R_1 X = E$ and then repeating (1). Reminder: for any matrix Y, $\operatorname{rank}(Y) = \operatorname{rank}(Y^t)$... why?)